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VOLUME XXXI NUMBER 6

THE PHYSICAL REVIEW

A JOURNAL OF EXPERIMENTAL AND THEORETICAL PHYSICS

CONDUCTED
WITH THE CO-OPERATION OF THE

AMERICAN PHYSICAL SOCIETY

BY

EDWARD L. NICHOLS,
ERNEST MERRITT, and FREDERICK BEDELL

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DECEMBER, 1910

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THE PHYSICAL REVIEW
LANCASTER, PA., AND ITHACA, N. Y.
BERLIN MAYER : AND MUELLER

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IN THE UNITED STATES

Annual Subscription $6.00 Single Copies $ .60

OUTSIDE THE UNITED STATES

Annual Subscription $6.50 Single Copies $ .65

Two volumes of THE PHYSICAL REVIEW are published annually, these volumes beginning in January and July, respectively, and containing six numbers each. Subscriptions should be sent to the publishers, THE PHYSICAL REVIEW, *41 North Queen Street, Lancaster, Pa.*, or *Ithaca, N. Y.*; or to Messrs. MAYER AND MULLER, *Berlin*. In order that subscriptions may terminate with the December issue, subscriptions for less than one year, terminating in December, will be taken at the yearly rate.

Previous to Volume V (July–December, 1897) THE PHYSICAL REVIEW was published in annual volumes, each containing six bi-monthly numbers, beginning with the July–August number, 1893.

A limited number of complete sets of the REVIEW will be supplied at the price of $3.00 per volume for all volumes, carriage extra. When they can be furnished without breaking a set, separate back volumes will be supplied at the same price, and single back numbers will be mailed to any address in the United States at the price of $.60 per copy, or to any address outside the United States at the price of $.65 per copy.

For advertising rates, address THE PHYSICAL REVIEW, *Ithaca, N. Y.* Change of copy in standing advertisements should be sent direct to the printers, THE NEW ERA PRINTING CO., *Lancaster, Pa.*

Correspondence relating to contributions should be addressed to the editors at *Ithaca, New York*. If an article submitted for publication in the REVIEW is to be published elsewhere, notice to that effect should be given. All manuscript submitted should be ready for the printer; the editors cannot assume responsibility for its correctness. Illustrations should be in black and white, and ready for reproduction. Curves should be plotted on plain paper with ruled black co-ordinates or on blue-lined cross-section paper. Cross-section paper with lines of any other color should be avoided.

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Entered at the post-office, at Lancaster, Pa., as second class matter.

Volume XXXI. *December, 1910.* *Number 6*

THE

PHYSICAL REVIEW.

ABSOLUTE FORMULÆ FOR THE MUTUAL INDUCTANCE OF COAXIAL SOLENOIDS.

By G. R. Olshausen.

1. Two absolute formulæ for the mutual inductance of coaxial solenoids have recently appeared, one by Cohen,[1] expressed in terms of elliptic integrals of Legendre and a second by Nagaoka[2] involving the $\wp$-function and σ-function of Weierstrass. The computation of any given case of mutual inductance by Cohen's formula necessitates the use of Legendre's tables of elliptic integrals, while a computation by Nagaoka's formula does not depend on the use of these tables, but on the introduction of the ϑ-functions of Jacobi.

A third formula, due to Kirchhoff, was published for the first time by Coffin.[3] This formula, which is of the same type as that of Cohen, was found, however, to be incorrect.

It is the object of this paper to derive a general expression from which the formulæ of Cohen and Nagaoka, as well as a correct form of Kirchhoff's formula may be obtained.

The mutual inductance of any two circuits is given by the expression

$$M = \int\int \frac{ds\,ds_1}{r} \cos\epsilon. \tag{1}$$

[1] L. Cohen, An Exact Formula for the Mutual Inductance of Coaxial Solenoids, Bull. Bur. Stand., 3, p. 295, 1907.

[2] H. Nagaoka, Note on the Mutual Inductance of Coaxial Coils, Mathematico-Physical Soc., Tokyo, Proc., 4, p. 192, 1907. Also l. c., p. 279, 1908.

[3] J. G. Coffin, Construction and Calculation of Absolute Standards of Inductance, Bull. Bur. Stand., 2, p. 125, 1906.

Applying this expression to two coaxial solenoids, it is easy to deduce

(2) $$M=4\pi nn'\,[I_1-I_2-I_3+I_4],$$ [1]

where n and n' are the respective numbers of turns per unit of length of the two coils, and the quantities in the brackets are integrals of the form

(3) $$I=a_1^2a^2\int_0^{\pi}\frac{\sin^2\psi d\psi}{a_1^2+a^2-2a_1a\cos\psi}\sqrt{a_1^2+a^2+c^2-2a_1a\cos\psi}.$$

In this a_1 and a are the respective radii of the coils and c has the following values:

$$c=d+l+l_1 \text{ for } I=I_1,$$
$$c=d+l_1-l \text{ for } I=I_2,$$
$$c=d+l-l_1 \text{ for } I=I_3,$$
$$c=d-l-l_1 \text{ for } I=I_4,$$

where d is the distance between the centers of the coils and $2l_1$ and $2l$ are their respective lengths.

To evaluate the above integral, let

$$\cos\psi=x,$$

then

1) $$I=\frac{\beta^{\frac{1}{2}}}{4}\int_{+1}^{-1}\frac{(x-1)(x+1)\left(x-\frac{\gamma}{\beta}\right)}{x-\frac{\alpha}{\beta}}\frac{dx}{\sqrt{(x-1)(x+1)\left(x-\frac{\gamma}{\beta}\right)}},$$

where

(5) $$\alpha=a_1^2+a^2,\quad \beta=2a_1a,\quad \gamma=a_1^2+a^2+c^2.$$

In order to reduce this integral to the normal form of Weierstrass, we let

(6) $$x-\gamma/\beta=m(s-e_1),\quad x-1=m(s-e_2),\quad x+1=m(s-e_3)$$

and determine the quantities e_1, e_2 and e_3 in such a way that their sum is zero.

m is a parameter, which can be determined by imposing upon it any arbitrary condition.

[1] See L. Cohen, l.c., p. 297.

From (4) and (6) we obtain

$$(7)\quad I=\frac{(m\beta)^{\frac{1}{2}}}{8}\int_{e_2}^{e_3}\frac{4(s-e_1)(s-e_2)(s-e_3)}{(s-e_3)-\frac{1}{m}\left(\frac{\alpha}{\beta}+1\right)}\frac{ds}{\sqrt{4(s-e_1)(s-e_2)(s-e_3)}}.$$

Now let

$$\mathfrak{p}u=\mathfrak{p}(u\,|\,\omega_1,\omega_3)^1$$

be the elliptic function which is determined by e_1, e_2 and e_3 and put

$$(8)\qquad s=\mathfrak{p}u\quad\text{and}\quad\mathfrak{p}w=e_3+\frac{1}{m}\left(\frac{\alpha}{\beta}+1\right).$$

We shall assume that ω_1 and ω_3/i are real and positive quantities. We have then

$$du=\frac{ds}{\sqrt{4(s-e_1)(s-e_2)(s-e_3)}}$$

and

$$4(s-e_1)(s-e_2)(s-e_3)=4\mathfrak{p}^3u-g_2\mathfrak{p}u-g_3=(\mathfrak{p}'u)^2,$$

where

$$(9)\qquad g_2=12e_3^2-4(e_1-e_3)(e_2-e_3),\quad g_3=4e_1e_2e_3.$$

On substituting in (7), we get

$$I=\frac{(m\beta)^{\frac{1}{2}}}{8}\int_{\omega_1}^{\omega_3}\frac{4\mathfrak{p}^3u-g_2\mathfrak{p}u-g_3}{\mathfrak{p}u-\mathfrak{p}w}\,du.$$

Expanding the integrand of this expression and remembering that

$$(\mathfrak{p}'w)^2=4\mathfrak{p}^3w-g_2\mathfrak{p}w-g_3,$$

we obtain

$$I=\frac{(m\beta)^{\frac{1}{2}}}{8}\int_{\omega_2}^{\omega_3}\left\{4\mathfrak{p}^2u+4\mathfrak{p}w\cdot\mathfrak{p}u+4\mathfrak{p}^2w-g_2+\frac{(\mathfrak{p}'w)^2}{\mathfrak{p}u-\mathfrak{p}w}\right\}du.$$

Integrating term for term, we have

$$(10)\quad I=\frac{(m\beta)^{\frac{1}{2}}}{2}\left[\mathfrak{p}w\cdot\eta_1+\left(\frac{g_2}{6}-\mathfrak{p}^2w\right)\omega_1+\frac{\mathfrak{p}'w}{2}\left\{\eta_1w-\omega_1\frac{\sigma'}{\sigma}(w)+n\pi i\right\}\right],$$

where n is an integer due to the many-valuedness of a logarithm. A method for determining n will be given later.

[1]The notation employed in this paper is that of the "Formeln und Lehrsaetze zum Gebrauche der elliptischen Functionen. Nach Vorlesungen und Aufzeichnung des Herrn K. Weierstrass, bearbeitet und herausgegeben von H. A. Schwarz."

Equation (10) enables us to evaluate the integrals I_1, I_2, I_3 and I_4 in all cases. In order to do this, it is only necessary to express the quantities in brackets in terms of the constants of the problem and to choose such a form for the parameter m as will facilitate the computation as much as possible.

2. As the number n will be seen to depend on the value of w which satisfies the known values of $\wp w$ and $\wp' w$, it is advantageous at this point to evaluate these quantities.

From (6) we find

$$e_1 - e_3 = \frac{1}{m}\left(1 + \frac{\gamma}{\beta}\right), \quad e_1 - e_2 = \frac{1}{m}\left(\frac{\gamma}{\beta} - 1\right), \quad e_2 - e_3 = \frac{2}{m},$$

(11)

$$e_1 = \frac{1}{m}\frac{2}{3}\frac{\gamma}{\beta}, \quad e_2 = \frac{1}{m}\left(1 - \frac{\gamma}{3\beta}\right), \quad e_3 = -\frac{1}{m}\left(1 + \frac{\gamma}{3\beta}\right).$$

From (5), (8) and (11) we get

$$\wp w = \frac{1}{m}\left\{\frac{3\alpha - \gamma}{3\beta}\right\} = \frac{1}{m}\frac{2(a_1^2 + a^2) - c^2}{6a_1 a} \tag{12}$$

and

$$\begin{cases} \wp w - e_1 = \dfrac{1}{m}\left\{\dfrac{\alpha - \gamma}{\beta}\right\} = \dfrac{-c^2}{2ma_1 a}, \\ \wp w - e_2 = \dfrac{1}{m}\left\{\dfrac{\alpha - \beta}{\beta}\right\} = \dfrac{(a_1 - a)^2}{2ma_1 a}, \\ \wp w - e_3 = \dfrac{1}{m}\left\{\dfrac{\alpha + \beta}{\beta}\right\} = \dfrac{(a_1 + a)^2}{2ma_1 a}. \end{cases} \tag{13}$$

Since

$$(\wp' w)^2 = 4(\wp w - e_1)(\wp w - e_2)(\wp w - e_3),$$

we have from (5) and (13)

$$\frac{\wp' w}{2} = \pm\sqrt{\frac{(\alpha^2 - \beta^2)(\alpha - \gamma)}{m^3\beta^3}} = \pm\frac{i(a_1^2 - a^2)c}{(m\beta)^{\frac{3}{2}}}. \tag{14}$$

In this equation we assume that $a_1 > a$ and that m and c are real and positive, *i. e.*, the absolute value of c is always to be employed.

It is easily seen that the value of $\wp w$ is such that

$$e_1 > \wp w > e_2,$$

for we have from (5), (8) and (11)

$$e_1 = \frac{1}{3m\beta}\,2\gamma = \frac{1}{3m\beta}\,2(a_1^2 + a^2 + c^2),$$

$$\wp w = \frac{1}{3m\beta}\,(3\alpha - \gamma) = \frac{1}{3m\beta}\Big[\,2(a_1^2 + a^2) - c^2\,\Big],$$

$$e_2 = \frac{1}{3m\beta}\,(3\beta - \gamma) = \frac{1}{3m\beta}\Big[\,2(a_1^2 + a^2) - \Big\{\,3(a_1 - a)^2 + c^2\,\Big\}\Big].$$

Since $\wp w$ is real and comprised between the values e_1 and e_2, w must have either the form

$$w = (2p+1)\,\omega_1 + (2q+t)\omega_3 \tag{15}$$

or

$$w = (2p+1)\omega_1 + (2q+1+t)\omega_3, \tag{16}$$

where p and q are any positive or negative integers, including zero, and t is a real positive quantity less than unity. If w is of the first form, $\wp' w$ is purely imaginary and positive. If w has a value corresponding to the second form, $\wp' w$ is purely imaginary and negative.

3. The integer n, which enters into the expression for I, can be determined for any given value of w as follows:

We have

$$-\int_{\omega_2}^{\omega_3} \frac{\wp' w}{\wp u - \wp w}\,du = \Big[\,\log\frac{\sigma(u+w)}{\sigma(u-w)} - 2u\,\frac{\sigma'}{\sigma}(w)\Big]_{\omega_2}^{\omega_3}$$

$$= 2\eta_1 w - 2\omega_1\,\frac{\sigma'}{\sigma}(w) + 2n\pi i.$$

If in the u-plane we start with any given value of w and let it vary along a straight line until the nearest value of w having the form

$$w_0 = (2p+1)\omega_1 + 2q\omega_3$$

is reached, the values of the definite integral and of the expression

$$2\eta_1 w - 2\omega_1\,\frac{\sigma'}{\sigma}(w)$$

can only change continuously; the number n therefore must remain

unchanged during this variation of w. When w has the value w_0, the definite integral equals zero, since $\mathfrak{p}'w$ is then zero, and

$$2\eta_1 w_0 - 2\omega_1 \frac{\sigma'}{\sigma}(w_0) = 2q\pi i.$$

We have therefore

$$n = -q.$$

4. By means of the addition theorem of the function $\frac{\sigma'}{\sigma}(u)$, equation (10) can be transformed into another form from which Cohen's expression for I can be derived.

We have

$$\frac{\sigma'}{\sigma}(u-v) = \frac{\sigma'}{\sigma}(u) - \frac{\sigma'}{\sigma}(v) + \frac{1}{2}\frac{\mathfrak{p}'u + \mathfrak{p}'v}{\mathfrak{p}u - \mathfrak{p}v}.$$

If we let

$$u = w \quad \text{and} \quad v = \omega_\lambda,$$

we obtain

$$\frac{\sigma'}{\sigma}(w-\omega_\lambda) = \frac{\sigma'}{\sigma}(w) - \eta_\lambda + \frac{1}{2}\frac{\mathfrak{p}'w}{\mathfrak{p}w - e_\lambda}, \tag{17}$$

where

$$\lambda = 1, 2 \text{ or } 3.$$

By means of (17) and the relation

$$\eta_1\omega_3 - \omega_1\eta_3 = \tfrac{1}{2}\pi i, \tag{18}$$

we can easily find three new expressions for I, but, for reasons which will appear later, we shall only consider the case $\lambda = 3$.

If $\lambda = 3$, we have

$$\eta_1 w - \omega_1\frac{\sigma'}{\sigma}(w) = \eta_1(w-\omega_3) - \omega_1\frac{\sigma'}{\sigma}(w-\omega_3) + \frac{\omega_1}{2}\frac{\mathfrak{p}'w}{\mathfrak{p}w - e_3} + \tfrac{1}{2}\pi i. \tag{19}$$

Substituting (19) in (10), we get

$$I = \frac{(m\beta)^{\frac{1}{2}}}{2}\left[\eta_1\mathfrak{p}w + \left\{\frac{g_2}{6} - \mathfrak{p}^2 w + \left(\frac{\mathfrak{p}'w}{2}\right)^2 \frac{1}{\mathfrak{p}w - e_3}\right\}\omega_1 + \frac{\mathfrak{p}'w}{2}\left\{\eta_1(w-\omega_3) - \omega_1\frac{\sigma'}{\sigma}(w-\omega_3) + (n+\tfrac{1}{2})\pi i\right\}\right]. \tag{20}$$

This equation leads to Cohen's formula, while from (10) Kirchhoff's formula can be deduced.

If in equation (10) we put

$$w = \omega_1 + w_1 i,$$

where w_1 fulfils the condition

$$0 < w_1 < \omega_3/i,$$

then w is of the form (15), the value of n is zero and $\mathfrak{p}'w$ *is positive.*[1]

Since we also have

$$\frac{\sigma'}{\sigma}(\omega_1 + w_1 i) = \eta_1 + \frac{\sigma_1'}{\sigma_1}(w_1 i),$$

the expression (10) becomes

$$I = \frac{(m\beta)^{\frac{1}{2}}}{2}\left[\eta_1 \mathfrak{p}w + \left\{\frac{g_2}{6} - \mathfrak{p}^2 w\right\}\omega_1 + \frac{\mathfrak{p}'w}{2}\left\{\eta_1 w_1 i - \omega_1 \frac{\sigma_1'}{\sigma_1}(w_1 i)\right\}\right]. \tag{21}$$

If in (20) we let w have such a value that

$$w - \omega_3 = \omega_1 + w_2 i,$$

or

$$w = \omega_2 + w_2 i,$$

where

$$0 < w_2 < \omega_3/i,$$

then w is of the form (16), n is -1 and $\mathfrak{p}'w$ is negative. Since we also have

$$\frac{\sigma'}{\sigma}(w - \omega_3) = \eta_1 + \frac{\sigma_1'}{\sigma_1}(w_2 i)$$

equation (20) becomes

$$I = \frac{(m\beta)^{\frac{1}{2}}}{2}\left[\eta_1 \mathfrak{p}w + \left\{\frac{g_2}{6} - \mathfrak{p}^2 w + \left(\frac{\mathfrak{p}'w}{2}\right)^2 \frac{1}{\mathfrak{p}w - e_3}\right\}\omega_1 + \frac{\mathfrak{p}'w}{2}\left\{\eta_1 w_2 i - \omega_1 \frac{\sigma_1'}{\sigma_1}(w_2 i) - \frac{1}{2}\pi i\right\}\right]. \tag{22}$$

5. In order to transform (21) and (22) into the corresponding formulæ of Kirchhoff and Cohen, we must find the relations between

[1] Nagaoka has given $\mathfrak{p}'w$ the wrong sign in this case; however, in his computation $\mathfrak{p}'w$ has the positive sign.

the integrals of Legendre and the quantities which enter into these equations.

If in the incomplete integral of the first kind to the modulus k

$$F(\varphi, k') = \int_0^{\phi} \frac{d\varphi}{\sqrt{1 - k'^2 \sin^2 \varphi}},$$

we let

$$\varphi = 90 - \psi,$$

we get

$$F(\varphi\ k') = \frac{1}{k} \int_{\pi/2-\phi}^{\pi/2} \frac{d\psi}{\sqrt{1 + \left(\frac{k'}{k}\right)^2 \sin^2 \psi}}.$$

where

$$k^2 + k'^2 = 1.$$

Changing the variable again, by letting

$$t = \sin^2 \psi$$

we obtain

$$F(\varphi, k') = \frac{i}{k'} \int_{\cos^2 \phi}^{1} \frac{dt}{\sqrt{4t(t-1)\left(t + \frac{k^2}{k'^2}\right)}}. \tag{23}$$

To transform this expression into the normal form of Weierstrass, let

$$t + k^2/k'^2 = m'(s - e_3), \quad t = m'(s - e_2), \quad t - 1 = m'(s - e_1), \tag{24}$$
$$e_1 + e_2 + e_3 = 0,$$

m' being a parameter. From these equations we find

$$e_1 - e_2 = \frac{1}{m'}, \quad e_1 - e_3 = \frac{1}{m'k'^2}, \quad e_2 - e_3 = \frac{k^2}{m'k'^2},$$
$$k^2 = \frac{e_2 - e_3}{e_1 - e_3}, \quad k'^2 = \frac{e_1 - e_2}{e_1 - e_3} \tag{25}$$

From (23), (24) and the second equation of (25) we have

$$F(\varphi, k') = i\sqrt{e_1 - e_3} \int_{\ }^{e_1} \frac{ds}{\sqrt{4(s - e_1)(s - e_2)(s - e_3)}}, \tag{26}$$

where the value of the lower limit is easily determined from the second equation of (24).

We have

$$s = \frac{t}{m'} + e_2 = \frac{\cos^2 \varphi}{m'} + e_2 = \cos^2 \varphi (e_1 - e_2) + e_2. \tag{27}$$

It is evident that for real values of φ, s is always real and comprised between e_2 and e_1.

If we now let

$$s = \wp u = \wp(u \mid \omega_1, \ \omega_3)$$

be the elliptic function belonging to the quantities e_1, e_2 and e_3, as determined above, we have

$$du = \frac{ds}{\sqrt{4(s - e_1)(s - e_2)(s - e_3)}},$$

and

$$F(\varphi, k') = i\sqrt{e_1 - e_3}\int_u^{\omega_1} du = i\sqrt{e_1 - e_3}(\omega_1 - u), \tag{28}$$

the lower limit u being determined by (27) for any given value of φ.

Since $s = \wp u$ is real and comprised between e_1 and e_2, we may put

$$u = \omega_1 + u_1 i,$$
$$0 < u_1 < \omega_3/i,$$

so that (28) becomes

$$F(\varphi, k') = u_1\sqrt{e_1 - e_3}. \tag{29}$$

If φ equals $\pi/2$, $F(\varphi, k')$ becomes a complete integral to the modulus k', *i. e.*,

$$F(k') = \frac{\omega_3}{i}\sqrt{e_1 - e_3}.$$

In the same manner it can be shown that the incomplete integral of the second kind to the modulus k'

$$E(\varphi, k') = \int_0^{\phi} \sqrt{1 - k'^2 \sin^2 \varphi}\, d\varphi,$$

reduces to

$$E(\varphi, k') = \frac{i}{\sqrt{e_1 - e_3}}\left[e_3(u - \omega_1) + \frac{\sigma'}{\sigma}(u) - \eta_1 \right],$$

$$= \frac{i}{\sqrt{e_1 - e_3}}\left[\frac{\sigma_1'}{\sigma_1}(u_1 i) + e_3 u_1 i \right]. \tag{30}$$

From this we find for the complete integral to the modulus k'

$$E(k') = \frac{i}{\sqrt{e_1 - e_3}} [\eta_3 + e_3\omega_3].$$

The expressions for the complete integrals of the first and second kind to the modulus k may be deduced in the same manner, as that in which the corresponding ones to the modulus k' were obtained. We have

$$F(k) = \omega_1\sqrt{e_1 - e_3}, \tag{31}$$

$$E(k) = \frac{1}{\sqrt{e_1 - e_3}} \left[\eta_1 + e_1\omega_1 \right]. \tag{32}$$

6. If in equation (21) we now express w_1, $\frac{\sigma_1'}{\sigma_1}(w_1 i)$, ω_1 and η_1 in terms of elliptic integrals of Legendre and g_2, $\wp w$, $\wp' w$, e_1 and e_3 in terms of the constants of the problem, we obtain the correct form of Kirchhoff's formula.

If after having fixed the parameter m, we assign to e_1, e_2 and e_3 in (25) the values given in (11), the moduli k and k' and the parameter m' will be definitely determined. If we also let

$$u = w = \omega_1 + w_1 i, \quad s = \wp w,$$

the value of the amplitude φ of the incomplete integrals can be determined from (27).

From (29), (30), (31) and (32) the following relations can be easily be deduced:

$$\left\{ \begin{aligned} w_1 &= \frac{1}{\sqrt{e_1 - e_3}} F(\varphi, k') \\ -\frac{\sigma_1'}{\sigma_1}(w_1 i) &= i\sqrt{e_1 - e_3}E(\varphi, k') + \frac{ie_3}{\sqrt{e_1 - e_3}} F(\varphi, k'), \\ \eta_1 &= \sqrt{e_1 - e_3}\left[E(k) - \frac{e_1}{e_1 - e_3} F(k) \right], \\ \omega_1 &= \frac{1}{\sqrt{e_1 - e_3}} F(k). \end{aligned} \right. \tag{33}$$

Substituting these values in (21), we get

$$I=\frac{(m\beta)^{\frac{1}{2}}}{2}\left[\sqrt{e_1-e_3}\cdot \wp w\cdot E(k)+\frac{1}{\sqrt{e_1-e_3}}\left\{\frac{g_2}{6}-\wp w(\wp w+e_1)\right\}F(k)\right.$$
$$\left.-\frac{i\wp' w}{2}\{F(k)[F(\varphi,k')-E(\varphi,k')]-E(k)F(\varphi,k')\}\right]. \tag{34}$$

The value of $g_2/6$ in this equation can be obtained from (9), which, by substituting from (11) and (5), becomes

$$\frac{g_2}{6}=\frac{2}{9m^2}\left[\frac{\gamma^2}{\beta^2}+3\right]=\frac{2}{9m^2}\left[\left(\frac{a_1^2+a^2+c^2}{2a_1a}\right)^2+3\right]. \tag{35}$$

From (5), (11), (12) and (35) we easily find

$$\wp w\sqrt{e_1-e_3}=\frac{1}{(m\beta)^{\frac{1}{2}}}\left[\frac{3\alpha-\gamma}{3}\right]\sqrt{\gamma+\beta}$$
$$=\frac{1}{(m\beta)^{\frac{1}{2}}}\frac{2(a_1^2+a^2)-c^2}{3}\sqrt{(a_1+a)^2+c^2} \tag{36}$$

and

$$\left\{\frac{g_2}{6}-\wp w(\wp w+e_1)\right\}\frac{1}{\sqrt{e_1-e_3}}=\frac{1}{(m\beta)^{\frac{1}{2}}}\left[\frac{\gamma^2+2\beta^2-3\alpha^2}{3\sqrt{\gamma+\beta}}\right]$$
$$=\frac{1}{(m\beta)^{\frac{1}{2}}}\left[\frac{c^4+2c^2(a_1^2+a^2)-2(a_1^2-a^2)^2}{3\sqrt{(a_1+a)^2+c^2}}\right]. \tag{37}$$

Substituting (14), taken with the positive sign, (36) and (37) in (34), we obtain

$$I=\tfrac{1}{2}\left[\frac{2(a_1^2+a^2)-c^2}{3}\sqrt{(a_1+a)^2+c^2}\,E(k)\right.$$
$$+\frac{c^4+2c^2(a_1^2+a^2)-2(a_1^2-a^2)^2}{3\sqrt{(a_1+a)^2+c^2}}F(k) \tag{38}$$
$$\left.+(a_1^2-a^2)c\{F(k)[F(\varphi,k')-E(\varphi,k')]-E(k)F(\varphi,k')\}\right],$$

which is the correct value of I for Kirchhoff's formula.

The amplitude φ may be obtained from (27). We have

$$\cos^2\varphi=\frac{\wp w-e_2}{e_1-e_2}$$

or

$$\sin^2\varphi=1-\frac{\wp w-e_2}{e_1-e_2},$$

and by (5), (11) and (13) this becomes

$$\sin^2\varphi = \frac{\gamma-\alpha}{\gamma-\beta} = \frac{c^2}{(a_1-a)^2+c^2}. \tag{39}$$

The moduli k and k' are obtained from (5), (11) and (25). We have

$$k^2 = \frac{4a_1a}{(a_1+a)^2+c^2}, \quad k'^2 = \frac{(a_1-a)^2+c^2}{(a_1+a)^2+c^2}. \tag{40}$$

7. If we now let

$$u = w-\omega_3 = \omega_1+w_2 i,$$

where

$$0<w_2<\omega_3/i,$$

and substitute the corresponding values of (33) in (22), we obtain

$$\begin{aligned} I = \frac{(m\beta)^{\frac{1}{2}}}{2}\Big[& \sqrt{e_1-e_3}\cdot \mathfrak{p}w\cdot E(k) + \frac{1}{\sqrt{e_1-e_3}}\left\{\frac{g_2}{6} - \mathfrak{p}w(\mathfrak{p}w+e_1)\right. \\ & \left. + \left(\frac{\mathfrak{p}'w}{2}\right)^2 \frac{1}{\mathfrak{p}w-e_3}\right\} F(k) \\ & - \frac{i\mathfrak{p}'w}{2}\left\{ F(k)[F(\varphi,k')-E(\varphi,k')] - E(k)F(\varphi,k') + \frac{\pi}{2}\right\}\Big]. \end{aligned} \tag{41}$$

In this equation $\mathfrak{p}'w/2$ is to be taken with its negative sign.

The only quantities in this equation, which have not been expressed in terms of the constants of the problem, are the coefficient of $F(k)$ of the second term in the square brackets and the amplitude φ.

From (5), (11), (13) and (14) we get

$$\begin{aligned} \frac{1}{\sqrt{e_1-e_3}}\left(\frac{\mathfrak{p}'w}{2}\right)^2 \frac{1}{\mathfrak{p}w-e_3} &= \frac{1}{(m\beta)^{\frac{1}{2}}}\frac{(\alpha-\beta)(\alpha-\gamma)}{\sqrt{\gamma+\beta}} \\ &= -\frac{1}{(m\beta)^{\frac{1}{2}}}\frac{(a_1-a)^2c^2}{\sqrt{(a_1+a)^2+c^2}}. \end{aligned}$$

Adding this to (37) and reducing, we get

$$\begin{aligned} & \frac{1}{\sqrt{e_1-e_3}}\left\{\frac{g_2}{6} - \mathfrak{p}w(\mathfrak{p}w+e_1) + \left(\frac{\mathfrak{p}'w}{2}\right)^2 \frac{1}{\mathfrak{p}w-e_3}\right\} \\ & = \frac{1}{(m\beta)^{\frac{1}{2}}}\left[\frac{c^4-(a_1^2-6a_1a+a^2)c^2-2(a_1^2-a^2)^2}{3\sqrt{(a_1+a)^2+c^2}}\right]. \end{aligned} \tag{42}$$

Substituting (36), (42) and (14), the latter taken with the negative sign, in (41), we finally have

$$I = \frac{1}{2}\Big[\frac{2(a_1^2+a^2)-c^2}{3}\sqrt{(a_1+a)^2+c^2}E(k)$$

$$+\frac{c^4-(a_1^2-6a_1a+a^2)c^2-2(a_1^2-a^2)^2}{3\sqrt{(a_1+a)^2+c^2}}F(k) \tag{43}$$

$$-(a_1^2-a^2)c\left\{F(k)[F(\varphi,k')-E(\varphi,k')]-E(k)F(\varphi,k')+\frac{\pi}{2}\right\}\Big],$$

which is the expression V in Cohen's formula.

It remains to find the amplitude φ for this expression. If in (27) we let

$$s=\wp(w-\omega_3),$$

and subtract e_3 from both sides of the equation, we obtain

$$\wp(w-\omega_3)-e_3=\cos^2\varphi(e_1-e_2)+(e_2-e_3)=(e_1-e_3)-(e_1-e_2)\sin^2\varphi.$$

Now

$$\wp(w-\omega_3)-e_3=\frac{(e_3-e_1)(e_3-e_2)}{\wp w-e_3},$$

and consequently

$$\sin^2\varphi=\frac{e_1-e_3}{e_1-e_2}\frac{\wp w-e_2}{\wp w-e_3},$$

which by (11), (13) and (5) becomes

$$\sin^2\varphi=\frac{\gamma+\beta}{\gamma-\beta}\frac{\alpha-\beta}{\alpha+\beta}=\frac{(a_1^2-a^2)^2+c^2(a_1-a)^2}{(a_1^2-a^2)^2+c^2(a_1+a)^2}. \tag{44}$$

The values of k^2 and k'^2 are of course the same as those in Kirchhoff's formula.

8. By letting λ equal 1 and 2 successively in (17) two additional expressions for I can be derived, but the value of the amplitude φ becomes complex in these cases. Nothing new is obtained by letting

$$v=-\omega_\lambda,$$

instead of

$$v=\omega_\lambda,$$

as was done on page 622 in deriving Cohen's formula.

9. The values of g_2, η_1, ω_1 and w depend on the value of m. If, for example, we let

$$\frac{(m\beta)^{\frac{1}{4}}}{2} = 2a_1a,$$

so that

$$m = \left(\frac{2}{a_1a}\right)^4,$$

and we then compute the above quantities, we obtain Nagaoka's values. His expression for I is obtained from (10) by giving m this value and letting n equal zero. A convenient value of m for computing I from (21) is the one for which $e_1 - e_3 = 1$. We shall give the special formulæ for this case later.

10. In addition to the expressions (11), (12), (13), (14), (35) and (40), two sets of formulæ are necessary for the computation of I from (21), one set for negative and another set for positive values of e_2. The series of the first set are more convergent when $e_2 < 0$, while those of the second set converge more rapidly when $e_2 > 0$.

From (5) and (11) we see that

$$e_2 \gtrless 0 \quad \text{when} \quad 6a_1a \gtrless a_1^2 + a^2 + c^2.$$

In the following formulæ all radicals are to be considered as positive quantities.

Formulæ for $e_2 < 0$.

Let

$$l = \frac{\sqrt[4]{e_1 - e_3} - \sqrt[4]{e_1 - e_2}}{\sqrt[4]{e_1 - e_3} + \sqrt[4]{e_1 - e_2}} = \frac{1 - \sqrt{k'}}{1 + \sqrt{k'}} \tag{45}$$

and

$$h = \tfrac{1}{2}l + 2(\tfrac{1}{2}l)^5 + 15(\tfrac{1}{2}l)^9 + 150(\tfrac{1}{2}l)^{13} + \cdots,$$

then ω_1 can be calculated from either of the two expressions

$$\omega_1 = \frac{2\pi}{(\sqrt[4]{e_1 - e_3} + \sqrt[4]{e_1 - e_2})^2}(1 + 2h^4 + 2h^{16} + \cdots)^2,$$

$$\omega_1 = \frac{\pi}{2\sqrt{e_1 - e_3}}(1 + 2h + 2h^4 + 2h^9 + \cdots)^2. \tag{46}$$

To determine η_1 we have

$$\eta_1\omega_1 = \frac{\pi^2}{12}\,\frac{1 - 3^3h^2 + 5^3h^6 - 7^3h^{12} + \cdots}{1 - 3h^2 + 5h^6 - 7h^{12} + \cdots}. \tag{47}$$

To find the value of w_1 which satisfies the values of $\wp w$ and $\wp' w$ given in (12) and (14), we write

$$\Omega_{0,1} = (\tfrac{1}{2})^2 l^4 + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 l^8 + \left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2 l^{12} + \cdots$$

$$\Omega_{0,2} = \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 l^8 + \left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2 l^{12} + \cdots$$

$$\Omega_{0,3} = \left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2 l^{12} + \cdots$$

and let

$$\wp w = s, \quad \frac{\sqrt[4]{e_1 - e_3}\sqrt{s - e_2} - \sqrt[4]{e_1 - e_2}\sqrt{s - e_3}}{\sqrt[4]{e_1 - e_3}\sqrt{s - e_2} + \sqrt[4]{e_1 - e_2}\sqrt{s - e_3}} = lt, \tag{48}$$

then

$$w_1 = \frac{\omega_1}{\pi} \log \text{nat}\,(\sqrt{t^2 - 1} - t) - \frac{2\sqrt{t^2 - 1}}{(\sqrt[4]{e_1 - e_3} + \sqrt[4]{e_1 - e_2})^2}\left\{\Omega_{0,1} t + \tfrac{2}{3}\Omega_{0,2} t^3 + \frac{2\cdot 4}{3\cdot 5}\Omega_{0,3} t^5 + \cdots\right\} \tag{49}$$

Instead of (48) and (49) the following two formulæ may also be employed:

$$\frac{\sqrt{e_1 - s} - \sqrt[4]{e_1 - e_3}\sqrt[4]{e_1 - e_2}}{\sqrt{e_1 - s} + \sqrt[4]{e_1 - e_3}\sqrt[4]{e_1 - e_2}} = lt', \tag{50}$$

$$w_1 = \frac{\omega_1}{\pi} \log \text{nat}\,\frac{1}{h(\sqrt{t'^2 - 1} - t')} + \frac{2\sqrt{t'^2 - 1}}{(\sqrt[4]{e_1 - e_3} + \sqrt[4]{e_1 - e_2})^2}\left\{\Omega_{0,1} t' + \tfrac{2}{3}\Omega_{0,2} t'^3 + \frac{2\cdot 4}{3\cdot 5}\Omega_{0,3} t'^5 \cdots\right\}. \tag{51}$$

If the absolute value of t is less than that of t', it will in general be better to use (49) to compute w_1, in the contrary case it is better to employ (51).

It can be shown that t and t' are always negative and that their absolute values are always greater than unity.

If we now put

$$v_1 = \frac{w_1}{2\omega_1}, \quad z = e^{\pi v_1},$$

then

$$\eta_1 w_1 i - \omega_1 \frac{\sigma_1'}{\sigma_1}(w_1 i)$$

$$= \frac{\pi i}{2}\left\{\frac{(z-z^{-1})+3h^2(z^3-z^{-3})+5h^6(z^5-z^{-5})\cdots}{(z+z^{-1})+h^2(z^3+z^{-3})+h^6(z^5+z^{-5})\cdots}\right\} \tag{52}$$

Formulæ for $e_2>0$

$$l_1 = \frac{\sqrt[4]{e_1-e_3}-\sqrt[4]{e_2-e_3}}{\sqrt[4]{e_1-e_3}+\sqrt[4]{e_2-e_3}} = \frac{1-\sqrt{k}}{1+\sqrt{k}}, \tag{53}$$

$$h_1 = \tfrac{1}{2}l_1 + 2(\tfrac{1}{2}l_1)^5 + 15(\tfrac{1}{2}l_1)^9 + 150(\tfrac{1}{2}l_1)^{13} + \cdots,$$

$$\frac{\omega_3}{\pi i} = \frac{2}{(\sqrt[4]{e_1-e_3}+\sqrt[4]{e_2-e_3})^2}(1+2h_1^4+2h_1^{16}+\cdots)^2,$$

or

$$\frac{\omega_3}{\pi i} = \frac{1}{2\sqrt{e_1-e_3}}(1+2h_1+2h_1^4+2h_1^9+\cdots)^2,$$

$$\omega_1 = \frac{\omega_3}{\pi i}\log\text{nat}\left(\frac{1}{h_1}\right), \tag{54}$$

$$\eta_1 = \frac{\pi i}{2\omega_3}\left[1-\tfrac{1}{6}\log\text{nat}\left(\frac{1}{h_1}\right)\cdot\frac{1-3^3h_1^2+5^3h_1^6-7^3h_1^{12}+\cdots}{1-3h_1^2+5h_1^6-7h_1^{12}+\cdots}\right]. \tag{55}$$

The formulæ which are necessary to determine w_1 in this case are:

$$\mathfrak{L}_{1,1} = \left(\frac{1}{2}\right)^2 l_1^4 + \left(\frac{1\cdot3}{2\cdot4}\right)^2 l_1^8 + \left(\frac{1\cdot3\cdot5}{2\cdot4\cdot6}\right)^2 l_1^{12} + \cdots,$$

$$\mathfrak{L}_{1,2} = \left(\frac{1\cdot3}{2\cdot4}\right)^2 l_1^8 + \left(\frac{1\cdot3\cdot5}{2\cdot4\cdot6}\right)^2 l_1^{12} + \cdots.$$

$$\mathfrak{L}_{1,3} = \left(\frac{1\cdot3\cdot5}{2\cdot4\cdot6}\right)^2 l_1^{12} + \cdots,$$

$$\frac{\sqrt{s-e_3}-\sqrt[4]{e_1-e_3}\sqrt[4]{e_2-e_3}}{\sqrt{s-e_3}+\sqrt[4]{e_1-e_3}\sqrt[4]{e_2-e_3}} = l_1 l_1', \tag{56}$$

$$\varphi_1 = \arctan\frac{l_1'}{\sqrt{1-l_1'^2}}, \quad -\frac{\pi}{2}\leqq\varphi\leqq\frac{\pi}{2}, \tag{57}$$

$$w_1 = \frac{\omega_3}{i}\left(\frac{1}{2} - \frac{\varphi}{\pi}\right) + \frac{2\sqrt{1-t_1'^2}}{(\sqrt[4]{e_1-e_3}+\sqrt[4]{e_2-e_3})^2}$$

(58)

$$\times\left\{\mathfrak{L}_{1,\,1}t_1' + \tfrac{3}{8}\mathfrak{L}_{1,\,2}t_2'^3 + \frac{2\cdot 4}{3\cdot 5}\mathfrak{L}_{1,\,3}t_1'^5 + \cdots\right\}.$$

Now let

$$v = \frac{w_1 i}{2\omega_3},$$

then

$$\eta_1 w_1 i - \omega_1 \frac{\sigma_1'}{\sigma_1}(w_1 i)$$

(59)

$$= i\pi v - \frac{\pi\omega_1}{\omega_3}\frac{2h_1\sin 2v\pi - 4h_1^4\sin 4v\pi + 6h_1^9\sin 6v\pi - \cdots}{1 - 2h_1\cos 2v\pi + 2h_1^4\cos 4v\pi - 2h_1^9\cos 6v\pi + \cdots}.$$

Formulæ corresponding to (48) and (49) could be given for this case, but the values $l_1 t_1$ and t_1 are complex and make the computation troublesome.

t_1' in the above formulæ may be either positive or negative; its absolute value is always less than unity.

In many cases l and l_1 are small quantities, so that the second terms of (49), (51) and (58) can be omitted. The approximate expressions for z and v are then quite simple.

We have from (49)

$$z^2 = \sqrt{t^2-1} - t, \tag{60}$$

from (51)

$$z^{-2} = h(\sqrt{t'^2-1} - t'), \tag{61}$$

and from (58)

$$v = \tfrac{1}{4} - \frac{\varphi}{2\pi}. \tag{62}$$

11. If we let

$$m = 1 + \frac{\gamma}{\beta} = \frac{(a_1+a)^2+c^2}{2a_1 a}, \tag{63}$$

so that

$$e_1 - e_3 = 1, \quad e_2 - e_3 = k^2, \quad e_1 - e_2 = k'^2,$$

$$s - e_1 = \frac{-c^2}{(a_1+a)^2+c^2}, \quad s - e_2 = \frac{(a_1-a)^2}{(a_1+a)^2+c^2}, \quad s - e_3 = \frac{(a_1+a)^2}{(a_1+a)^2+c^2},$$

some of the formulæ of the last paragraph simplify somewhat. The other quantities which are necessary to calculate I from (21) have the following forms in this case:

$$m\beta = (a_1+a)^2+c^2, \tag{64}$$

$$s = \mathfrak{p}w = \frac{1}{3}\,\frac{2(a_1^2+a^2)-c^2}{(a_1+a)^2+c^2}, \tag{65}$$

$$\frac{\mathfrak{p}'w}{2} = \frac{i(a_1^2-a^2)c}{[(a_1+a)^2+c^2]^{\frac{3}{2}}}, \tag{66}$$

$$\frac{g_2}{6} = \frac{2}{9}\left[\left\{\frac{a_1^2+a^2+c^2}{(a_1+a)^2+c^2}\right\}^2 + 12\left\{\frac{a_1 a}{(a_1+a)^2+c^2}\right\}^2\right]. \tag{67}$$

Since $e_1-e_3=1$, e_2 can be expressed in terms of m alone. We easily find

$$e_2 = \frac{1}{3}\left(\frac{4}{m}-1\right),$$

from which we see that

$$e_2 \lesseqgtr 0 \quad \text{when} \quad m \lesseqgtr 4.$$

When $c=0$ and $a_1=a$ we have from (63)

$$m=2,$$

which is the minimum value of m. This is one of the values of m when the two coils coincide, so that we have self-inductance instead of mutual inductance. As a very convenient expression for computing the self inductance of a single layer coil has been given by Nagaoka,[1] it is not necessary to give formulæ for self inductance here.

12. Cohen's as well as Kirchhoff's formula involves c^4, which is inconveniently large when one or both coils are long, or when there is a great distance between their centers. Kirchhoff's formula has the further disadvantage that, when c is large compared to (a_1+a), the arguments φ and θ of Legendre's tables are both near 90°, so that $E(\varphi, k')$ and especially $F(\varphi, k')$ cannot be determined with sufficient accuracy, unless terms of the fourth and even fifth order are taken into account in the interpolation.

[1] H. Nagaoka, Note on the Self-inductance of Solenoides, *l. c.*, p. 314, 1908.

13. The case computed by Cohen, where

$$d=0, \quad 2l=200, \quad 2l_1=20, \quad a=10, \quad a_1=15,$$

when calculated by Kirchhoff's formula, gave

$$M=4\pi nn' \, 6212.9,$$

while formula (21), when (61) was used and m was so chosen that $e_1-e_3=1$, gave

$$M=4\pi nn' \, 6213.77,$$

seven place logarithms having been employed. The values of M obtained by Cohen and Nagaoka were

$$M=4\pi nn' \, 6213.4$$

and

$$M=4\pi nn' \, 6213.51,$$

respectively.

A computation by the series given for $e_2>0$, which do not converge very rapidly for this case, gave

$$M=4\pi nn' \, 6213.63.$$

The results of the computation by (21) and (61) are given on the following page.

	$a_1=15,\ a=10,\ c=110.$	$a_1=15,\ a=10,\ c=90.$
$\log \mathfrak{p}w = \log \frac{1}{3}\frac{2(a_1^2+a^2)-c^2}{(a_1+a)^2+c^2}$	$9_n.4770264$	$9_n.4542696$
$\frac{g_2}{6} = \frac{2}{9}\left[\left(\frac{a_1^2+a^2+c^2}{(a_1+a)^2+c^2}\right)^2 + 3\left(\frac{2a_1a}{(a_1+a)^2+c^2}\right)^2\right]$	0.2122382	0.2079914
$\log \frac{\mathfrak{p}'w}{2i} = \log \frac{(a_1^2-a^2)c}{[(a_1+a)^2+c^2]^{\frac{3}{2}}}$	7.9813160	8.1400044
$\log (m\beta)^{\frac{1}{2}} = \log [(a_1+a)^2+c^2]^{\frac{1}{4}}$	6.1569867	5.9111481
$k^2 = \frac{4a_1a}{(a_1+a)^2+c^2}$	0.0471513	0.0687679
$k'^2 = 1-k^2$	0.9528487	0.9312321
$\log l = \log \frac{1-\sqrt{k'}}{1+\sqrt{k'}}$	7.7808415	7.9496612
$\log h = \log \left(\frac{l}{2}+\cdots\right)$	7.4798115	7.6486312
$\log \omega_1 = \log \frac{2\pi}{(1+\sqrt{k'})^{\frac{1}{2}}}(1+\cdots)^2$	0.2013483	0.2038209
$\log \eta_1 = \log \frac{\pi^2}{12\omega_1}\frac{1-9h^2\cdots}{1-3h^2\cdots}$	9.7136753	9.7110910
$\sqrt{e_1-s} = \sqrt{\frac{c^2}{(a_1+a)^2+c^2}}$	0.9751330	0.9635180
$\log lt' = \log \frac{\sqrt{e_1-s}-\sqrt{k'}}{\sqrt{e_1-s}+\sqrt{k'}}$	7.8164572	7.9856940
$-t'$	1.085465	1.086508
$\log z = \log [h(\sqrt{t'^2-1}-t')]^{-\frac{1}{4}}$	1.1709451	1.0860006
$\log \left\{\frac{(z-z^{-1})+3h^2(z^3-z^{-3})+\cdots}{(z+z^{-1})+h^2(z^3+z^{-3})+\cdots}\right\}$	9.9977951	9.9967327
$\log \left\{\ \right\}\frac{\mathfrak{p}'w}{2}\cdot\frac{\pi i}{2} = \log \frac{\mathfrak{p}'w}{2}\left\{\eta_1 w_1 i - \omega_1 \frac{\sigma_1'}{\sigma_1}(w_1 i)\right\}$	$8_n.1752310$	$8_n.3328570$
$\mathfrak{p}w\cdot\eta_1$	-0.1551321	-0.1463391
$\omega_1\left(\frac{g_2}{6}-\mathfrak{p}^2w\right)$	0.1943994	0.2030302
$\frac{\mathfrak{p}'w}{2}\left\{\eta_1 w_1 i - \omega_1\frac{\sigma_1'}{\sigma_1}(w_1 i)\right\}$	-0.0149703	-0.0215207
Sum $= \Sigma$	0.0242970	0.0351704
$\log \Sigma$	8.3855527	8.5461773
$\log 2I = \log (m\beta)^{\frac{1}{2}}\cdot\Sigma$	4.5425394	4.4573254
$2I$	34877.02	28663.25

$$2(I_1 - I_2) = 2(I_4 - I_3) = 34877.02 - 28663.25 = 6213.77$$
$$M = 4\pi nn'6213.77$$

WORCESTER POLYTECHNIC INSTITUTE,
August 23, 1910.

FURTHER STUDIES OF THE FIREFLY.

By Herbert E. Ives.

UNDER the title "The Light of the Fire-fly"[1] the present writer, in conjunction with Dr. W. W. Coblentz, has described an investigation of the visible radiation of one variety of firefly (*Photinus Pyralis*). The chief difference between that investigation and preceding ones (for instance that of Professor Langley) lay in the application of photography with color-sensitive plates. By this means the difficulties imposed by the faint and intermittent character of the light were largely overcome. Measurements of energy distribution, practically impossible by visual means, were made, and the character and extent of the visible radiation were determined from photographic exposures which integrated the effect of the fire-fly's flashes during periods of several hours. The visible radiation was found to consist of a continuous band of yellow-green light extending from .51μ to .67μ, with a maximum at .57μ, the wave-length of maximum visual sensibility for high illuminations and likewise of maximum energy in the solar spectrum, as received through the atmosphere.

If this narrow band in the visible region of the spectrum constitutes the total radiation, it would appear that the fire-fly—as far as its radiated energy is concerned—is the most efficient of known illuminants. Its (radiant) efficiency was calculated to correspond to nearly fifty candles per watt, little short of what it would have been had the light been quite monochromatic of wave-length .57μ.

The object of the present paper is to discuss more fully the grounds for believing the firefly's visible radiation to be its total radiation. Experiments in the infra-red and ultra-violet regions of the spectrum were recently made to investigate this important question and the results are here described.

[1]Transactions of the Illuminating Engineering Society, October, 1909, p. 675. Also Bulletin of the Bureau of Standards, vol. 6, no. 3.

The only experiments heretofore made outside the visible region are those of Langley and Very, recorded in their memoir "The Cheapest Form of Light."[1] The bolometer measurements there described showed clearly the very small quantity of heat radiated by the insects, as compared with any artificial source, for a given quantity of light. These results are generally interpreted as showing that, aside from body heat, the visible radiation is the only radiation. Study of the paper referred to, however, shows that one possibility in the character of the radiation was not investigated, namely, that infrared radiation of a quantity comparable with the visible may exist between the regions of visible and "body" radiation. In order that this deficiency in Langley's work may be clearly understood, his experiments will be briefly outlined, with his conclusions and with comments to indicate the reasons for believing a more conclusive test necessary.

It is first pointed out that there will be a certain amount of heat radiated, due to the body temperature of the insect. This heat, corresponding to a temperature below 50° C., will lie in the region of waves much longer than those ordinarily accompanying visible radiation in the production of light by high temperatures. This radiant heat will be distinguished from radiation other than body radiation by its inability to pass through glass, which is transparent to wave-lengths of less than 3μ. The body radiation may therefore be separated from that of shorter wave-length than 3μ by the interposition of a piece of glass. This obviates the necessity which would otherwise exist of forming a spectrum, with consequent great loss of light.

With the most sensitive bolometer at his command, Langley then made the following experiment. The radiation from an insect was allowed to fall on the bolometer surface. A small but real galvanometer deflection was obtained, representing the excess of total heat radiation from the insect over that from a metal plate at 25° C. used as the background. A sheet of glass was then interposed, *which cut off all the observed heat* (*i. e.*, there was no observed galvanometer deflection).

The next experiments were comparisons of the energy radiated

[1]Smithsonian Miscel. Collections, no. 1258.

by the firefly and by a flame. A "nearly non-luminous Bunsen flame," much fainter than the insect, was compared with the latter. The total radiation from the insect gave one-fourth of a division deflection, the flame 177.5 divisions. The interposition of glass cut down the flame radiation to 14.5 divisions, still very much greater than the total radiation from the insect, for a smaller quantity of light. Subsequently measurable indications of insect radiation through the glass (shorter than 3μ) were obtained, and those showed that "flames whose intrinsic brilliancy is nearly comparable to that of a point below the middle of the candle flame, and whose total brilliancy is as exactly as possible comparable to that of the insect, give several hundred times the heat of the latter, even if we consider only that quality of heat which is found above 3μ."

The point to which our particular attention should be directed here is this: In every case where the insect radiation above 3μ was allowed to fall upon the bolometer, *not only the infra-red but also the visible radiation was acting.* Hence, when the very small deflections were observed, the question was left entirely open as to whether they came from the visible radiation, or from some radiation between the limit of the visible spectrum (about $.75\mu$) and 3μ. The comparison with an equally luminous flame did not settle this question, for similar results would follow were there, for instance, a second emission band in the insect spectrum of equal strength between $.75\mu$ and 3μ. Data are lacking as to the exact energy distribution in this "nearly non-luminous flame," but it is quite safe to say that the energy in the infra-red region was so very much more than in the visible that the roughly qualitative measure of "several hundred times" would have applied equally well had there been firefly radiation of equal quantity to the visible, but lying between $.75\mu$ and 3μ. In other words, the test which was lacking was a measurement of the energy in the radiation from 3μ down, with and without a screen opaque to the *visible radiation.* Had there been marked deflection with the total radiation, measurable deflection with that from 3μ down, and none with the visible and that greater than 3μ eliminated, then the restriction of the radiation to the visible region and to body heat would have been

established. It should be noted that Langley did not specifically state the latter to be the case, his emphasis being laid more on the smallness of the heat radiation as compared with ordinary illuminants. In the diagram representing the firefly's radiation however, it is given as entirely in the visible region.

The firefly's actual efficiency as a light source is dependent to a large degree on the radiation being confined to the visible region. If there should be found infra-red of quantity comparable with the visible, the firefly, while still a very efficient source, would not be, as usually supposed, the example of an ideally efficient light produced by nature.

The chief interest in the study of the firefly lies perhaps in the possibility of producing light artificially by a similar process. The exact character of this light is therefore a matter of importance. It is important for us to know whether nature has produced merely a very efficient light or has produced the most efficient light possible. In the latter case we might conclude with some reason that a process of selection has been gone through and so that it is not beyond the realms of possibility that we may discover similar means of producing light, not so efficient, but more suitable to our use by virtue of the more pleasing color of the light obtained.

At the beginning of the present firefly season the writer did not have available facilities for repeating the very delicate bolometric work of Langley; therefore some other means of continuing the investigation was sought. In view of the extreme difficulty of making radiometric observations on the faint and intermittent light of the firefly, by far the best method for continuing the investigation of its radiation would be a photographic one, were such available. But the best color-sensitive plates at present known are sensitive only as far as the deep red. Another method which has been employed in the study of the infra-red spectrum occurred to the writer as a possible means of carrying the study of the firefly spectrum through part of the region between the visible and the body radiation. This is the phosphor-photographic method, employed by Draper, Lommel, Dahms and others. In brief, it consists of this: Phosphorescence, which is excited in various substance by exposure to short waves (blue, violet or ultra-violet),

is destroyed by exposure to longer waves (orange, red, infra-red). Thus, a surface of Balmain's paint or of Sidot blende, excited to phosphorescence and then exposed in a spectrograph, will have areas of reduced brightness wherever long wave energy has fallen upon it. If this surface is then laid on a photographic plate for a short period a permanent record is obtained on the plate after development.

In order to test the applicability of this method to the light of the firefly certain information had to be obtained as follows:

1. What phosphorescent substance combines the longest duration of phosphorescence with the greatest susceptibility to the extinguishing action of long waves?

2. What quantity of energy is necessary to produce extinction?

3. What is the spectral extent of this extinguishing action?

4. What is the relative response to extinguishing action in different parts of the spectrum?

5. What is the period after excitation during which extinction of phosphorescence is produced?

6. Is the extinguishing action integrative, *i. e.*, are a large number of short exposures (such as the flashes of the firefly) equivalent to a continuous exposure of the same total length?

The first question was answered by the choice of Sidot blende, phosphorescent zinc sulphide. This substance continues to glow sufficiently for photography for many hours after excitation, and exhibits the property of extinction by infra-red more strongly probably than any other known substance. A small quantity, prepared by Verneuil, was secured from Eimer and Amend and used throughout the investigation.

As to the quantity of energy necessary to produce the extinguishing action, no exact quantitative measurement was made. Such a measurement would have had no value without a knowledge of the quantity of energy radiated from the firefly. If, for instance, it were known that a ergs per second were radiated in the visible spectrum by the firefly, the problem would be to search for radiation of the same order of magnitude in the infra-red, and therefore a method to be applicable should be capable of detecting this quantity of energy. The visible radiation from the firefly being of

unknown quantity, this line of attack was not seriously considered. An idea of the high sensitiveness of the method was obtained from the rough observation that the light of an incandescent lamp through a ruby glass (absorbing the short waves which excite phosphorescence) at 10 cm. distance, sufficed to cause noticeable reduction of phosphorescence in a second.

The property of Sidot blende which made it suitable for this search for infra-red was learned upon the study of the spectral region in which extinction occurs. This region is not only in the infra-red but also in the visible, as far as the green. It follows then that the visible radiation of the insect should also extinguish phosphorescence. It was therefore thought that the test for the applicability of the method would be to determine whether the visible radiation of the firefly would cause reduction of phosphorescence. If so, then, provided the sensibility to extinction were of the same order of magnitude in the infra-red as in the visible, the method would be applicable. This led to investigation of questions 3 and 4.

The response to long-wave excitation has been found by Dahms and by Nichols and Merritt to extend from the green of the visible spectrum to 1.5μ. Part of the region in which we are interested would therefore be capable of investigation.

As to the relative response to extinguishing action in different parts of the spectrum, no suitable data were available. What is here required is the relative action at different wave-lengths for the same quantity of energy. The only quantitative determination of extinguishing action is that of Nichols and Merritt.[1] Using a Nernst glower as source, they took as a measure of extinguishing action the reciprocals of the relative times of decay of phosphorescence to a certain value with and without exposure to infra-red. Their measurements indicate a maximum of action in the infra-red of 1.3μ, with very slight action in the visible region and a minimum in the neighborhood of 1μ. As however the energy radiated by the Nernst glower is far greater in the infra-red than in the visible, the maximum of reducing action may be expected to lie to the shorter wave-length side of 1.3μ for an equal energy spectrum. In the absence of other data and of facilities for making a direct

[1] PHYS. REV., Vol. 25, p. 380.

determination, it was decided to use the data of Nichols and Merritt, making a reduction to a normal spectrum from a knowledge of the energy distribution of the Nernst glower. The latter was obtained from a curve by Coblentz[1] which is for a glower near normal consumption, as the one used by Nichols and Merritt presumably was. A further correction was then made for the dispersion of the quartz prism which was used to form the spectrum of the Nernst. This was calculated from data given by Kayser.[2] The original curves (two different days) as given by Nichols and Merritt, and the reduced curves are given in Fig. 1 plotted to equality at .7μ. No great accuracy can be claimed for the final curve, because of the indirect method of obtaining it. It shows however that the impression obtained from the original curves that the chief action is in the infra-red is apparently wrong, the real maximum of action lying at .7μ in the visible. The amount of action in the infra-red as far as 1.5μ appears however to be everywhere at least of the same order of magnitude as the action at .57μ, the wave-length of the visible firefly radiation. If, therefore, reducing action could be obtained by the visible radiation, it would show that the method is available for detecting infra-red radiation down to 1.5μ.

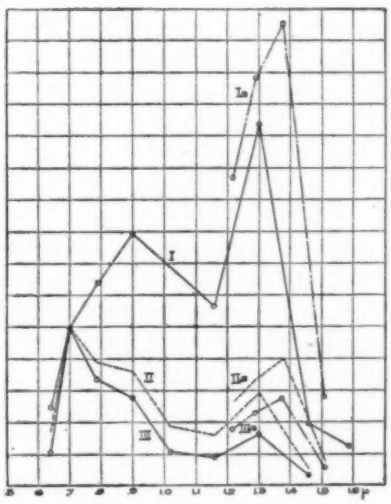

Fig. 1.

Extinguishing action of infra-red on the phosphorescence of sidot blende. I. Values obtained by Nichols and Merritt; II. Same corrected for energy distribution of Nernst glower; III. Further correction for prismatic dispersion.

Use of this data of Nichols and Merritt carries with it the assumption that their means of measuring sensibility to reducing action—by time of decay to a given value—is equivalent to a method

[1]Bulletin of Bureau of Standards, 4, p. 543.

[2]Handbuch der Spectroscopie, Vol. 4, p. 486.

measuring by quantities of energy. In other word, a small quantity of energy acting over a long period is equivalent to a large quantity over a short period. This assumption, which holds approximately for the photographic plate, was tested with a Sidot blende surface, by exposures of two, four and eight minutes in a quartz spectograph in which the slit openings for each exposure were inversely as the time of exposure. As nearly as could be judged, the photographed phosphorescence showed equal reducing action in all exposures. The use of the data as reduced appears therefore to be justified.

There is of course involved as well in using this data the assumption that the Sidot blendes used by Nichols and Merritt and in the present investigation are identical in their properties. This can be considered as probable, but could only be established by repeating their investigation.

The next question to be answered was the length of the period after excitation during which extinction of phosphorescence is produced. To test this a series of four holes was made in a metal plate, each hole covered with a photographic absorption screen of different density. A surface of Sidot blende, immediately after excitation by the light of a quartz mercury arc, was exposed for one minute through this plate to the radiation from an incandescent lamp with a ruby bulb at a distance of 25 cm. At intervals of one hour thereafter similar exposures were made on successively adjacent portions of the surface. After five such exposures the Sidot blende was left for an hour and then a photographic plate exposed upon it for two hours. This plate showed clearly that when the photographic exposure is made as long after excitation as six hours the extinguishing action due to infra-red exposure immediately after excitation is much less than that due to exposure after one hour, that the action is nearly uniform for about three hours, and that after three hours the effect again decreases. This was confirmed by further experiments. In order therefore to take the best advantage of the method, the Sidot blende should be excited some time before the exposure to infra-red, and the exposure to firefly light should not be continued (to advantage) longer than three hours.

The final question to be considered before actual work with fire-

flies was whether the phosphorescent Sidot blende acted toward a feeble and intermittent illumination as a photographic plate does; that is, will the action of the separate flashes be added? This is a question to be answered only by experiment, for it might well happen that a certain quantity of energy is necessary to start the extinguishing action, any quantity less than this having no effect. An artificial firefly was therefore constructed—a disc which could be rotated by a motor, the disc having a small opening near its circumference. A quartz spectrograph was set up and the slit and (incandescent lamp) light source adjusted to give by visual estimate about the intrinsic brightness of the firefly in the yellow. An exposure of two minutes was found sufficient to produce a reversed spectrum. The opening in the disc was then made 4° so that the exposure would be for one-ninetieth of the total time. Three hours of intermittent exposure through this disc would thus give a total exposure of two minutes. Such an exposure was therefore made, and on the same surface a continuous exposure of two minutes at the middle of the three-hour period. The resultant photograph showed as much action by the intermittent exposure as by the continuous one of the same total length, possibly more.

The result of these preliminary experiments was, therefore, to show the applicability of the method of phosphor-photography in detecting any firefly radiation down to 1.5μ, *provided* the visible radiation was strong enough to effect reversal. The method appears to be as suitable for firefly investigation in the infra-red as is the photographic method in the visible, granting that the conclusions hold which have been drawn as to relative sensibility to radiation of different wave-lengths.

The method having been tested, arrangements were made to try it on fireflies. This work had of necessity to be carried on in a more southern latitude that that of Cleveland, where the writer is situated, and where fireflies are small and scarce. The Physical Laboratory of the Johns Hopkins University was kindly placed at his disposal, and he spent several days there.

The first experiment consisted in exposing the phosphorescent surface (one-half hour after excitation) to the total light of the firefly. Two metal plates with 6 mm. diameter holes in their

centers were laid one over the other with a sheet of glass 1 mm. thick between. The lower plate was separated by a fraction of a millimeter from the Sidot blende.

An exposure of 20 minutes to the flashes of a single firefly was sufficient to produce observable reduction of phosphorescence. Exposure was discontinued after half an hour and a photograph was taken (Cramer Inst. Iso. plate) which showed with great clearness the reduction of the exposed area. This proved that sufficient energy was radiated by the firefly to render the method applicable. The time of exposure which sufficed showed too that the period of response to extinguishing action, about three hours, should be enough to detect a fractional part of the energy which produced the observed radiation.

The next experiment was to determine the location of this energy in the spectrum. A deep ruby glass, nearly opaque to the visible radiation but very transparent in infra-red, was substituted for the piece of clear glass. If the reduction of phosphorescence was caused chiefly or wholly by the visible radiation then there should be little or no reducing effect with the red glass. If on the contrary there is infra-red energy and if this is comparable in quantity to the visible radiation, reduction should occur even with the red glass.

Trial showed that an exposure of three hours and a half to firefly light through the red glass had no reducing effect that could be detected. Therefore it follows at once that there is no infra-red energy between 1.5μ and .75μ in quantity comparable with the visible radiation.

The visible and infra-red regions as far as 1.5μ having been covered by photography and phosphor-photography, attention was turned to another region of the spectrum, namely, the ultra-violet. The spectrogram obtained a year ago extended as far as .38μ and the shorter wave-length region was not investigated. Since then, however, R. W. Forsyth[1] reports finding ultra-violet radiation from luminescent bacteria, formerly considered as giving monochromatic green radiation. In view of this observation a search for short-wave energy in the radiation of the firefly seemed necessary.

[1]Nature, 83, p. 7, 1910.

A small Fuess quartz spectograph was employed, with Cramer spectrum plates. The dispersion in this instrument is such that the spectrum from .6μ to .2μ is about 4 cm. long, and with a quartz mercury arc photographic action is obtained as far as .216μ. With a wide slit an exposure of an hour was made to firefly light. The resultant negative showed strong action at .57μ, but no trace of action elsewhere. There is then no ultra-violet radiation of magnitude comparable with the visible.

These experiments therefore cover the spectrum from the extreme ultra-violet to the infra-red at 1.5μ. Between this point and the region opaque to glass lies an interval of 1.5μ, in which there may be radiated energy. It has been surmised that the outer skin of the insect is opaque to radiation of wave-length greater than 2μ and perhaps with the water contained to 1.5μ. Were this true it would follow that as far as the energy radiated from the insect is concerned the present investigation is sufficient to establish its restriction to the visible region and to "body heat." It was thought of interest to investigate the transmission of the abdominal horny skin of the firefly, and for this purpose a small piece of skin, about 1$\times$3 mm. in size, was removed from an insect and laid upon a glass microscope slide. To the eye it appeared somewhat cloudy, but quite transparent in parts, of a pale yellow color by transmission. This was first investigated by the phosphor-photographic method, with a quartz spectrograph and found to be free from absorption bands, or noticeable difference in absorption as far as 1.5μ. Investigation was then continued with a bolometer. Needless to say, the bolometric sensibility requisite is far less than would be necessary for measuring the fireflies' light. A linear bolometer and mirror spectrometer furnished with a fluorite prism were available. With the assistance of Dr. C. F. Lorenz, the energy distribution from a Nernst glower, bare, the same through a piece of glass microscope slide, and finally the energy as transmitted by the piece of skin, mounted on the glass slide, were measured. The ratio of the last to the second set of values so obtained gave the transmission of the firefly's skin. Measurements were made both with the skin dry and with it wet. It proved in its natural thickness not to be opaque for waves longer than 1.5μ. On the contrary, the trans-

mission showed a gradual increase from the visible region to beyond 3μ, being for this piece about half as much again in the extreme infra-red as on the edge of the visible. This shows that there yet remains to be investigated the region from 1.5μ to $3\ \mu$.

At the conclusion of these experiments on the nature of firefly radiation, some experiments were made with a view to learning a little more about the process of light emission involved. Although these results were practically negative, they may be of some interest because they were carried out to test a different theory of the firefly light than those usually held.

The firefly radiation is commonly supposed to be produced by some chemical process, probably a slow oxidation, this process being largely under the control of the insect. On the other hand, true phosphorescent substances derive their light from some external source; they store up the energy, to reëmit after excitation has ceased. In view of the large amount of incident radiation and the small amount emitted, this process is probably very inefficient.

Some time ago in making experiments on the effect of heat on phosphorescent substances, the writer was struck with the similarity of the flash given out by heated Sidot blende to the flash of the firefly. If for example a small pile of Sidot blende is laid on a thin metal plate, excited, and allowed to decay to a dull glow, application of a Bunsen burner to the plate causes a vivid flash of phosphorescent light. This may be repeated several times. Might not the light of the firefly be true phosphorescence from a phosphorescent substance of slow decay, whose emission of light is stimulated by heat or oxygen, as is true of certain phosphorescent substances? In this case the emitted light would be stored-up solar energy absorbed during the day, and the process of light production would not necessarily be efficient at all.

It is true that this theory would call for a powerfully phosphorescent material, capable of storing up probably a larger quantity of energy than any known. Certain peculiarities of fireflies seemed nevertheless to fit in with this idea; such as the fact that they are most active shortly after sunset and that later they are apparently scarcer and less active. While no great confidence was felt in this theory, it was thought worth while to apply several of the charac-

teristic tests of phosphorescence to the firefly, such as the influence of ultra-violet and infra-red radiation, and confinement in the dark. Experiments of this particular kind have, with one exception, not previously been made, and might, if not confirming the theory, at least add to our knowledge of the firefly's light. The one exception is the influence of heat. Experiments are cited which show an increase of light emission in fireflies, glow-worms, and such creatures, when warmed. This is similar to the action of heat on true phosphorescence, but is as well what one would expect if heat acts merely as an irritant or stimulant.

As already stated, these experiments were practically negative. The only effect on the living insects of exposure to strong light (arc, Cooper-Hewitt mercury arc) was to cause them to entirely cease to glow for a few minutes after exposure and return to a dark room. Ordinarily, in the dark, there are two small spots which emit a dull glow continuously independent of the characteristic flashing. Exposure to strong light extinguished these, as well as causing the flashes to cease. After a short interval of waiting and upon vigorous shaking the insects again glowed and flashed. Their response to light would therefore indicate control and apparent appreciation on their part that their own light is unnecessary under these circumstances. Ultra-violet (from an iron arc) and infra-red radiation (carbon arc and ruby glass) produced no visible effect on the living insects.

Two batches of insects were placed in glass vessels, furnished with moist grass and leaves and covered with wire screen. One vessel was kept in a dark room for twenty-four hours, the other kept in a light room where sunlight fell upon it during the afternoon. In the evening both lots were examined. There appeared to be no difference in their behavior. Both were less active than freshly caught insects, but the ones confined in darkness responded to shaking quite as well as the ones which had enjoyed the daylight. There was, therefore, no indication that previous exposure to light is necessary to enable the firefly to emit light.

Finally some of the abdominal material was spread on a glass and different parts of it exposed to the light of a bare carbon arc, to the arc light through a deep red glass and through a cyan blue

glass. Long exposure failed to show the slighest effect. Apparently, therefore, the only effect on the luminescent material of exposure to light is to be found in the living insects. In these the light is discontinued by some voluntary process or nervous reflex.

The results of this further study of the firefly may be here summarized: The spectrum of the firefly radiation has been studied from the ultra-violet at .2μ to the infra-red at 1.5μ. Photography was employed for the ultra-violet and visible regions, the extinction of phosphorescence for the infra-red. The only radiation found is the band at .57μ previously investigated. Apparently none of the same order of magnitude exists elsewhere in the region studied. The abdominal skin of the insect being here shown to be transparent certainly as far as 3μ, there remains for investigation the interval from 1.5μ to 3μ, if we agree to call all beyond 3μ "body heat." That all beyond 3μ is merely low temperature radiation still of course lacks experimental proof. The light of the firefly has been investigated for the properties of true phosphorescence, and found to respond to none of the characteristic tests, such as dependence upon previous exposure to light or extinction by long waves. The light appears to be under the control of the insect.

In conclusion it may be remarked that obviously the most satisfactory method of studying the whole region between body heat and visible radiation would be by a method sensitive to the whole interval. Probably more sensitive bolometers than that of Langley may now be constructed, and the measurements in question made. The opportunity, however, presented itself to try out the phosphorphotographic method as a means of detecting very faint radiation. It has proved well suited to that purpose, and, possessing the valuable integrating property of the photographic plate, is perhaps better adapted, within its range of sensitiveness, to detecting faint radiation than any radiometric method. More confidence may be felt in its failure to give evidence of infra-red radiation than in the failure of the bolometer of Langley, since the latter was practically incapable of detecting the visible radiation to which the phosphorphotographic method is sensitive to nearly the same degree as to infra-red.

The results of the present investigation may be said to render

still more probable the belief that the firefly produces only visible radiation, of high efficiency. There is no evidence that the light is due to anything other than a true production of light by some bio-chemical process, which, as Langley remarks, "we know nothing to prevent our successfully imitating."

PHYSICAL LABORATORY OF THE NATIONAL ELECTRIC LAMP ASSOCIATION,
CLEVELAND, OHIO.

THE DISCHARGE OF ELECTRICITY THROUGH GASES AT VARIOUS TEMPERATURES.

BY R. F. EARHART.

IN a previous paper,[1] some simple experiments were described wherein it was sought to determine the potentials required to produce and to maintain a luminous discharge through the air and other gases at different temperatures. The investigation referred to was limited to a temperature range lying between $-78°$ and $315°$ C.

It did not seem desirable to extend the temperature range without eliminating some of the objectionable features inherent in the form of apparatus used at that time. In the previous experiment the electrodes consisted of flat parallel disks contained in a chamber of limited dimensions but with no insulation which would prevent discharge occurring from the edges of the disks.

If a form of discharge chamber shown in Fig. 1 be used, after the pressure is reduced to a particular value, the discharge will not pass through the shortest distance between the plates, but will take a longer path by striking back along the lead wires.

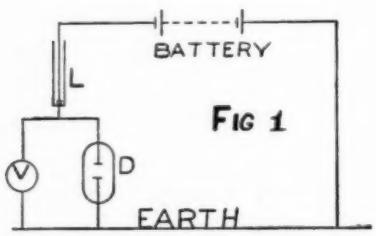

Fig. 1.

This is illustrated by the lecture experiment known as the Hittorf tube. Moreover observation of the luminous discharge shows that if the field is not uniform, the appearance of the discharge changes perceptibly with variation in pressure. In many cases the discharge is confined to the region between the disks. However, if discharge occurs from the edges of the disks it is not possible to state what quantity of gas functions in the discharge.

[1] PHYS. REV., Sept., 1909, p. 293.

After trying several forms of apparatus, the simplest one was adopted and is represented in Fig. 2. The container for the gas and connection with the pump and gauge consisted of hard glass, through which platinum wires were sealed. This was a section of bomb tube 18 mm. internal diameter and 12 cm. in length. The electrodes were of copper faced with a heavy platinum plate. These were held in place by an insulator *S* which served to insulate the edges of the electrodes and confine the discharge to a known volume of gas. Equalization of pressure was secured by

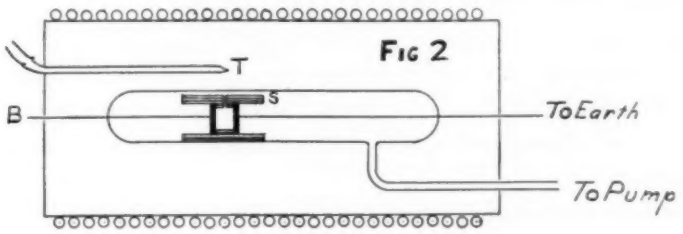

Fig. 2.

piercing the insulator with a tiny hole. The discharge chamber within this insulating support was 5 mm. long and 10 mm. in diameter.

This support for the electrodes was turned from a block of soap-stone secured from Messrs. Hammell and Gillespie of New York. This grade of soap-stone possesses qualities which make it especially desirable in a physical laboratory. In its natural state it tools beautifully, making it a simple matter to secure a special insulator of any desired form. Upon baking at a temperature of 1000° or thereabouts, its physical character changes. It becomes exceedingly hard, so that it is difficult to cut either with a file or an emery wheel. It possesses excellent insulating properties and so far as could be determined with a McLeod gauge, does not occlude any large quantity of gas when used at lower temperatures. The particular advantage, however, of this grade of stone over the common variety is its freedom from cracking and disintegration at high temperatures.

The electrical arrangement used for determining the potential required to produce the luminous discharge is very simple and is shown in Fig. 1.

A storage battery consisting of 600 cells has one terminal grounded. The other terminal is carried to one of the electrodes in the discharge chamber D through a liquid resistance L. The other electrode of the discharge chamber is earthed. A Weston voltmeter with a suitable multiplier V is placed in parallel with the discharge chamber. By decreasing the resistance in L, the potential registered by the voltmeter rises until a discharge occurs through D. This discharge is accompanied by a violent kick of the voltmeter needle. Upon decreasing the potential until the discharge ceases the cessation of the discharge is again indicated by the voltmeter.

The discharge tube and container were placed in an electric furnace which consisted of a porous battery jar wound with nickel wire. The whole was imbedded in a large box of loose magnesium packing.

The temperature was measured by means of a Pt. Pt-Rh thermo-couple T. The temperature thus determined is merely the temperature of the air-bath surrounding the discharge chamber. After the thermo-couple showed constancy for a half hour, it was assumed that the temperature within the furnace had become uniform. Frequently the temperature as shown by the thermo-couple was maintained constant for several hours. However a half-hour period and a 24-hour period of constancy gave fully consistent results.

With this form of apparatus it was possible to assume that the volume of the gas through which the discharge passed was limited to the volume of gas contained between the electrodes. Changes in the volume of the receptacle due to temperature changes fall within the limits of accuracy of the potential measurements. The volume of gas functioning during discharge being constant, it is possible then by measuring pressures and temperatures to determine the mass of gas operated upon under varying conditions of temperature and pressure.

The results obtained for air are represented graphically in Fig. 3. These apply to the potentials required to produce the luminous condition. It may be noted that the minimum potential required at the temperatures from 22° to 560° differ but slightly from one another. The potentials required to maintain luminosity in the

region of the critical pressure for all of these temperatures is approximately 345 volts, differing not more than 5 volts from this and seemingly independent of the temperature throughout this range.

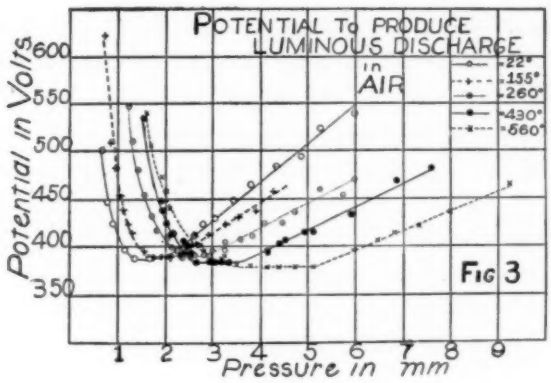

Fig. 3.

It was noted that for pressures less than the critical pressure the potentials for maintaining discharge increased when the pressures became so small as to require potentials larger than the minimum for the production of the discharge.

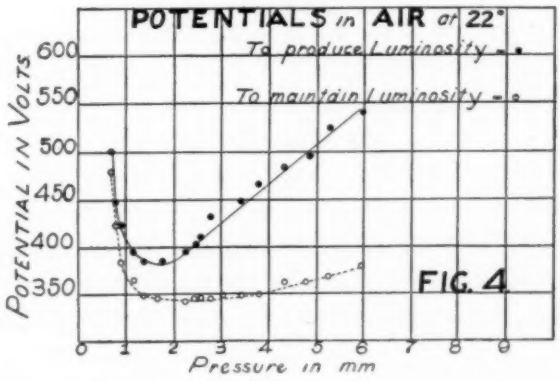

Fig. 4.

In Fig. 4 a typical pair of curves is shown, the upper curve indicating the potential required to produce luminosity, the lower curve the one required to maintain luminosity. The curve repre-

senting the potential required to maintain luminosity falls in the same place, viz., 345 volts for the region corresponding to the critical pressure and for temperature variations from 22° to 560°. This potential agrees approximately with what is known as the minimum spark potential in air as given by a number of observers. If this is the interpretation it would seem that the minimum spark potential was unaffected throughout this range.

Measurements were made for a temperature of 630° in air, the readings for the higher pressures falling where they were to be expected. However when the pressures were decreased to such values that one would anticipate the necessity of higher potentials to produce the discharge, no such change occurred, the graph representing such readings running parallel to the X axis. A study of this showed that a condition of temperature and pressure had been

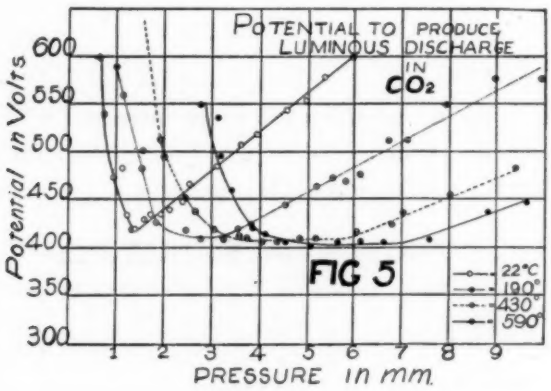

Fig. 5.

attained such that the discharge flashed outside the insulator, decreasing pressures merely changing the alternative path. This was found to be the case when the alternative path was about 3 cm. in length.

The same phenomenon occurred in the neighborhood of 430° for hydrogen. This was obviated for hydrogen by making the protecting hood 8 cm. in length and the readings for 460° obtained. This length did not prove effective for air above 600°. Figs. 5 and 6 represent the results secured for hydrogen and CO_2.

No stress is laid upon the purity of the gases. Hydrogen was prepared in a Kipp apparatus using C.P. platinitized zinc and H_2SO_4. It was afterwards passed through potassium permanganate and pyrogallic acid and a drier of phosphor pentoxide. It was deemed desirable to allow the gases to remain for some time in the heated tube before taking observations. With the insulator at times red hot and the electrodes in a similar condition, it would not be sur-

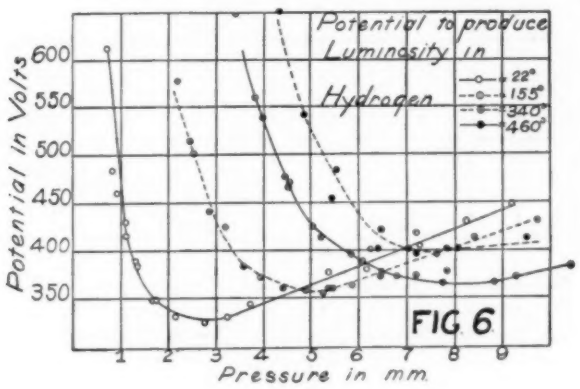

Fig. 6.

prising if foreign gases were occluded which would perceptibly affect the electrical properties of the gases. It was only possible to admit reasonably pure gases and proceed under the working conditions. In general, the curves for hydrogen show consistent values for each separate curve. There is, however, a possible lack of consistency in their agreement one with the other.

In Fig. 7 the readings represented in Fig. 4 are all reduced to show the relation between the mass of gas between the electrode and the potential required to produce the discharge through this gas. The volume of the gas through which the discharge passes being constant, the mass will vary directly as the pressure and inversely as the absolute temperature.

According to Paschen's law these values should fall on a line since the potential required for a discharge varies with the mass of the gas. The agreement between theory and the experimental results is indicated in the graph. The scale chosen is one which will tend to bring out discrepancies rather than for the purpose of pro-

ducing a smooth curve. Save in the portion of the curve where the gradient is very steep there are few deviations which exceed 15 volts from the mean. In the portion of the curve where the gradient is steep very small errors in the pressure measurement will produce great changes in the potential readings.

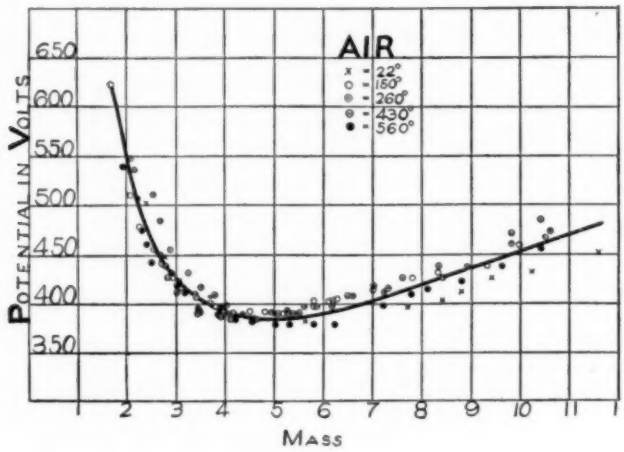

Fig. 7.

In Fig. 8 the results for CO_2 are reduced to the same basis. The agreement here is not so satisfactory. However the general trend of the values is similar to that indicated for air.

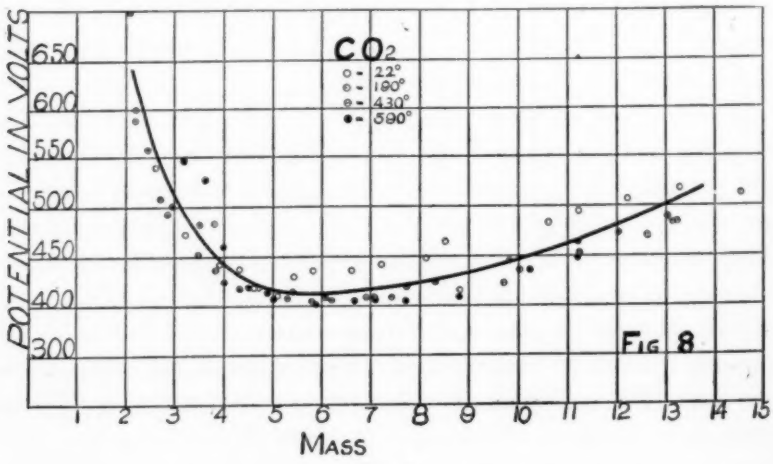

Fig. 8.

The values for hydrogen show the same characteristics; the agreement of values is not so satisfactory. In my opinion this is due to the introduction of impurities. The electrical properties of hydrogen seem to be greatly affected by traces of air or other gases. It was not possible to obtain measurements above 600° with this form of apparatus. This is due in part to the discharges striking around the protecting hood of the insulator, and in part due to the fact that the glass container softens at the higher temperatures and collapses in the softened condition.

It seemed possible to make some observations at still higher temperatures with a slightly different form of apparatus in order to determine whether potentials smaller than the ones discussed hitherto would ionize the gas.

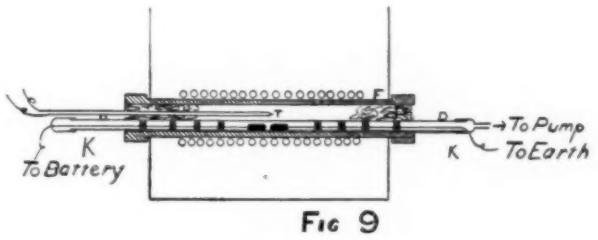

Fig. 9.

The form of apparatus used is indicated in Fig. 9. The furnace consisted of a large porcelain tube, *F*, 25 inches long and 1 inch in diameter. The middle portion of this was wound with nickel wire and imbedded in magnesium packing. The ends were allowed to project from the packing. A tube 36 inches long and one-fourth inch internal diameter, *D*, was introduced into the furnace tube. This contained two cylinders of Pt rounded at the ends, 24 mm. long and 4 mm. in diameter separated 1 cm.

Connection with these electrodes was made wth heavy Pt wires. In one series of experiments a quartz tube was used but did not prove satisfactory. This was perhaps due to the fact that the opaque quartz was used. This has the appearance of being drawn, the walls consisting of an immense number of capillary tubes with their axes in general parallel to the axis of the tube. In that case there is probably some communication between the tubes. Whether

or not the tube selected was faulty in construction, or this is common to the opaque quartz, it was impossible to maintain a constant pressure. A slow leak was invariably present. A tube of the Royal Berlin porcelain glazed inside and outside proved satisfactory. Spaced along the Pt lead wire, small porcelain insulators were placed. These were for the purpose of reducing convection currents. The tube in which the discharge occurred was placed symmetrically in the furnace tube with the thermo-couple T near the electrodes. The furnace tube was then plugged with asbestos fiber P. The lead wires were brought out through brass caps; the caps being sealed at the joints K with Khotinski cement. The projecting ends of the porcelain and the large surface of the brass caps permitted a sufficient air cooling to render this cement effective in maintaining a tight joint

One end of the tube was connected with the pump and gauge as in the previous experiment. In this case it is not possible to

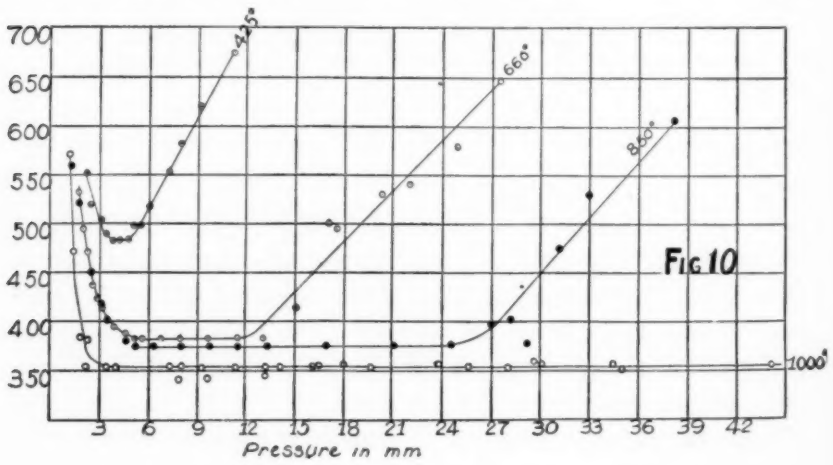

Fig. 10.

compare the quantities of gas which function when the discharge occurs under varying conditions. The least distance between the electrodes may constitute the path of the discharge but the alternative path may vary through wide limits. However it is not unreasonable to suppose that starting with high pressures the discharge takes the shortest discharge between the electrodes and with

decreasing pressure the discharge will continue in this manner until a critical pressure is reached when the discharge will strike back along the electrodes. When this condition is reached, decreasing the pressures will merely cause the path to change, the potential showing no variations. It was hoped that in this way some data might be obtained on the effect of temperature on the minimum of potentials. The results obtained are indicated in Fig. 10.

The values for the minimum potential at 660° are practically the same as for the range between 22° and 560°. The ones for 850° differ only 6 volts from this. The values for 1000° lie at 355 volts. The curve representing this latter case begins to rise at a pressure of 5 cm. but is not shown on the plate. The upward trend of the curves occurring at low pressures has no quantitative significance. It shows merely that the reduced pressures were effective in elevating the discharge potential.

At 660° and 850° the potentials required to maintain luminosity were similar to those secured for the lower temperatures.

At 1000° it had fallen to 332 volts. The variations at 1000° from those at lower temperatures are not large.

Summary.

The general conclusions may be stated briefly.

1. Potentials required to produce luminous discharge from 22° to 600° depend merely on the mass of gas between the electrodes.

2. Potentials required to maintain a discharge in air do not fall below 345° until temperature approaching 1000° C. are reached.

3. The minimum energy required to ionize a gas by collision depends to a very slight extent upon the temperature of the gas, for temperatures less than 1000°.

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ON THE QUESTION OF THE MOTION OF THE ETHER IN A STEADY ELECTROMAGNETIC FIELD.

By S. J. Barnett.

IF we admit the existence of the ether and the reality of Maxwell's stresses, the ether within an element of volume dv of an electromagnetic field devoid of matter will be acted upon by a force

$$dF = \frac{1}{c^2}\dot{s}dv,$$

where $\dot{s}$ denotes the time derivative of the Poynting vector and c the velocity of light. Hence, on the above hypothesis, the ether, when it becomes the seat of a steady electromagnetic field, must be set into persistent motion in the direction of the energy stream.

An investigation of the possible motion of the ether in the case of damped electric waves and some other cases not discussed here was made in the Cavendish Laboratory in 1897 by Messrs. Henderson and Henry,[1] but their paper escaped my notice until about a year ago. They used a Lodge interferometer, with a short path, and obtained no effect, the minimum velocities which they could have detected ranging from 11.5 meters/sec. to 200 meters/sec.

About eight years ago I planned to study this matter experimentally, with an interferometer, but facilities were not available. Within the last year, however, it has been practicable to undertake some work upon the subject, and a description of this work, which was completed in April, is given below. Although the investigation resembles in some respects that of Henderson and Henry, its undertaking was deemed advisable in view of the fact that the important case considered here was not examined by them, and because it was possible with the apparatus described here to attain a precision in measuring a possible velocity of the ether from fifty to one thousand times that attained by them.

[1] W. C. Henderson and J. Henry, Phil. Mag., vol. 44, 1897, p. 20.

The electromagnetic field investigated was produced between two horizontal parallel copper bars *HI*, and two similar bars *KJ*, connected at one end by copper strips *L* and *M*, as shown in Fig. 1. By means of two double-pole double-throw switches, not shown in the figure, the terminals of a direct-current dynamo could be connected to two of the bars near the end *H*, and the terminals of a rheostat to the other two near the end *K*; and the dynamo and rheostat connections could be interchanged by reversing the two switches simultaneously. With this arrangement the energy stream could be sent along the bars in either direction, *HIJK* or *KJIH*, at will. The copper bars were 0.3 cm. thick, 2.5 cm. wide, and over 2 meters long. The 2.5 cm. sides were vertical. The bars of each pair were about 9.5 mm. apart, and were blackened on their inner faces.

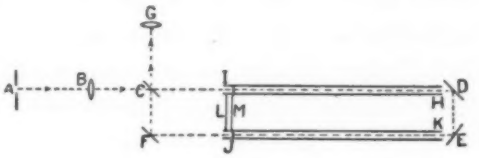

Fig. 1.

The state of motion of the ether in the electromagnetic field was investigated by means of an interferometer of the general type used by Michelson and Morley and also by Lodge in their experiments on moving matter. It was similar to the instrument used by Lodge[1] except that a vertical *slit* was used as the source of light, and that the fringes were observed with an eyepiece in the manner of Fresnel and not with a telescope. The apparatus was arranged as shown in Fig. 1. *A* was the vertical slit at one end of the collimator *AB*, *C* a half-silvered plane plate of glass, *D*, *E*, *F* three silvered plane glass mirrors, and *G* a positive eyepiece with vertical spider thread. Light from a 90° carbon arc lamp after passing through the collimator was incident in the direction *ABC* upon the plate *C*. Here it was divided into two parts, one of which traversed the course *CDEFCG*, the other the course *CFEDCG*, between the copper bars. When the proper adjustments were made, the two beams overlapped along *CG* and vertical interference fringes appeared in the eyepiece at *G*. The collimator and eyepiece were parts of a Dubosq spectrometer, which was mounted upon a pier. The glass plate *C* was mounted upon the prism

[1] O. J. Lodge, Roy. Soc. Phil. Trans., A, vol. 184, 1893, p. 727.

table in a frame provided with a screw for adjustment in altitude. The adjustment in azimuth was made by the tangent screw of the prism table. The first adjustment, which had to be made with great precision in order to obtain brilliant and clear fringes, was made in part by the imperfect screw and then perfected by moving small weights about on the prism table. The mirror F was mounted on the same pier, and the mirrors D and E were mounted on a slate shelf carried by a brick wall. The copper bars were mounted on a heavy wooden frame extending from the pier to the shelf. The mirrors were all ground by Petitdidier, and were excellent. The total length of the electromagnetic field traversed by each beam of light was about 425 cm. The presence of air in the field was of course a matter of no consequence.

If the velocity of the ether is denoted by u and the length of the path of light in the electromagnetic field by l, the retardation of one of the beams over the other is approximately

$$T = \frac{2ul}{c^2}.$$

In the experiments described here a piece of green glass was placed in front of the slit, and it was estimated that a lateral motion of a dark band through a distance as great as 1/50 or 1/60 the distance between the centers of two adjacent dark bands would have been detected. Moreover, observations were made on *reversing* the direction of the energy stream. Hence, the wave-length of the light used being taken as 5×10^{-5} cm. and its period therefore $5/3\times10^{-15}$ sec., we have, to determine the smallest velocity u which would have been detected, the equation

$$\frac{1}{2}\times\frac{1}{50}\times\frac{5}{3}\times 10^{-15}\text{ sec.} = \frac{2u\times 425\text{ cm.}}{9\times 10^{20}\ \frac{\text{cm.}^2}{\text{sec.}^2}};$$

which gives

$$u = 18\text{ cm./sec.}$$

In the first experiments currents between 40 and 50 amperes were used; and in the later ones currents between 80 and 100 amperes. The potential difference between the copper bars in the first case was about 105 volts, and in the second about 90 volts. The electric in-

tensity E in the two cases was thus about 110 volts/cm. and 100 volts/cm. The magnetic intensity H of the field was determined with a test coil and ballistic galvanometer. It was about 7.5 gausses for a current of 40 amperes, and 15 gausses for a current of 80 amperes. In neither case was a lateral shift observed on reversal. Hence it must be inferred that the ether did not move along the energy stream with a velocity as great as about 1/5 meter/sec.

In a recent paper, an abstract of which reached me while this investigation was in progress, E. Cunningham[1] has come to the conclusion that if electromagnetic momentum can be regarded as ether momentum, the ether, in the case considered here, must have a velocity which satisfies the equation

$$u = \frac{c^2 + u^2}{c} \frac{EH}{E^2 + H^2}.$$

This equation, in which E and H are expressed in the Gaussian system, has the solutions

$$u = c\frac{E}{H} \quad \text{and} \quad u = c\frac{H}{E}.$$

In the experiments described here, E was about $\frac{1}{3}$ electrostatic unit and H about 17 electromagnetic units when the current was about 90 amperes. For this case, therefore, the first of the above solutions gives the velocity of the ether as about one fiftieth that of light, while the second gives a velocity about fifty times that of light—both entirely discordant with the results of direct observation.

This investigation appears therefore to favor decidedly the principle of relativity.

I am indebted to Mr. Hall for instrumental work on this investigation, and to Mrs. Barnett for assistance in making the experiments.

NEW ORLEANS, 1910.

[1]E. Cunningham, Roy. Soc. Proceedings, Series A, Vol. 83, 1909, p. 109.

A SENSITIVE THERMOPILE.

BY B. J. SPENCE.

AMONG the various types of thermopiles for use in the study of spectral distributions and radiation in general, the Rubens thermopile, consisting of junctions of fine iron and constantin wire seems to be the most efficient. The line thermopile described by Rubens[1] consisted of twenty junctions of constantin and iron each 0.5 mm. in width and formed a line of 20 mm. in length. The thermo-electromotive force per junction as given by Rubens is 53×10^{-6} volts per degree difference of temperature between the hot and cold junctions, thus giving a total of 1.06×10^{-3} volts per degree difference of temperature between the twenty hot and twenty cold junctions. The resistance of the thermopile was 3.5 ohms.

Rubens using a galvanometer of 2.77×10^{-10} amperes, and a resistance of 5 ohms would have obtained a deflection of about 0.5 mm. for a difference of temperature of one millionth of a degree. No data were given for the sensibility of the thermopile-galvanometer combination in terms of a "candle at a distance of a meter" from the thermopile.

Rubens however does not claim for his thermopile an absolute sensibility equal to that of the bolometer. It has been the desire of the writer to construct a thermopile of an absolute sensibility equal to that of the best bolometer of equal area upon which to receive radiation, and hence possessing a much higher effective sensibility, effective in the sense that a bolometer and thermopile of equal absolute sensibility, the thermopile is the more steady. The bolometer, while a very efficient instrument in itself possesses a very serious drawback and difficulty attending its use, namely battery drift and unsteadiness.

The bolometer and thermopile are alike in that they are ex-

[1]Zeitschrift fur Instrumentenkunde, 18, 65.

tremely susceptible to minute changes in surrounding temperature unless properly encased. The thermopile as a rule possesses a difficulty which is foreign to the bolometer. The thermopile does not respond so quickly to thermal charges or return to its initial zero position as rapidly as does the bolometer, owing to its larger thermal capacity. Hence in constructing a thermopile, two things must be borne in mind—first it must possess a high sensibility, and second it must have a very small thermal capacity.

In order to attain the first it is necessary to obtain metals or alloys which possess high thermo-electromotive forces, and at the same time have comparatively low thermal capacity. It is likewise necessary that the metals or alloys be not too brittle, so that it becomes an unmanageable task to handle them.

The writer was referred to the work of Hutchins,[1] which gives a description of alloys of bismuth and tin and bismuth and antimony. The alloy of bismuth and tin contained approximately 5 per cent. tin, and the alloy of bismuth and antimony contained approximately 3 per cent. antimony. A single pair of junctions of these alloys gave a thermo-elctromotive force of 120×10^{-6} volts per degree difference of temperature over a wide range of average temperatures, *i. e.*, between the hot and cold junctions.

In order to prepare the alloys for use, they were melted and while in a liquid state pressed between two pieces of plate glass. This method of preparation gave them a smooth surface and almost any thickness desired could thus be obtained. Sheets of these alloys were selected which had a thickness of about 0.25 mm. They were then cut into bars about 2.5 mm. long and 0.5 mm. wide. The most convenient and economical method of procedure was to rule lines on the sheets of alloy with a knife, and then by a further use of the knife the bars could be readily cut.

The bars were then mounted on a piece of ivory about 22 mm. long and of slightly less width than the length of the bars of alloy. The mounting was so done that the cold junctions came on one edge of the ivory piece and the hot junctions on the other edge. The bars were insulated from each other by thin paper and shellac, the shellac serving a double purpose, namely, as an insulation and

[1]Am. Jour. Science, 48, 226.

also as an adhesive with which to bind the bars together and also to bind them to the ivory piece. As little shellac as possible was used owing to the difficulty it gave when the soldering of the junctions was attempted.

The ends of the rows of hot and cold junctions were filed with a very fine file and the junctions were properly made by means of a soft solder consisting of lead, tin and bismuth. The remaining flat side of the thermopile was then shellaced and covered with ivory of the same dimensions as the first.

By this means a pile of thirty to forty junctions was obtained having a resistance of 3.5 ohms to 5 ohms, according to the number of junctions. The pile was then mounted in a circular disc of hard rubber, having a thickness of about 1 cm. Copper leads were fixed into the disc and fine copper wires from the terminals of the pile wire soldered to these leads. The whole was then enclosed in a cylindrical water jacket open at the ends. The cylindrical jacket contained suitable blends to prevent convection currents, and stray radiation from affecting the thermopile.

For a description of the complete thermopile, one having 37 hot junctions will be chosen. It possessed a length of 21 mm. and a width of 0.5 mm. The bars were 2.5 mm. long. The thermal electromotive force per degree difference of temperature between hot and cold junctions for the complete thermopile was 4.4×10^{-3} volts, and had a resistance of 5 ohms. This thermopile connected to a galvanometer having a resistance of 3 ohms and a sensibility of 4×10^{-10} amperes would give, for a difference of temperature of one millionth of a degree, a galvanometer deflection of 1.4 mm. as compared to a deflection of .4 mm. given by the Rubens thermopile with a much more sensitive galvanometer.

A much more satisfactory way of determining the sensibility of the thermopile is to find the sensibility in terms of the "candle at a distance of a meter" and perhaps at the same time compare the deflection given by the thermopile with that of the bolometer of equal area, using the same galvanometer for each. The sensibility of the galvanometer in this case was 4×10^{-10} amperes. For a candle at a distance of 3 meters from the thermopile a deflection of 260 mm. was obtained, giving a deflection of 2,340 mm. at a

distance of one meter. A bolometer gave a deflection for a candle at a distance of a meter of 1100 mm. The bolometer had a rock salt window to protect it from air currents while the thermopile did not possess a window. The comparison in this case is not so important for it was evident that the bolometer did not possess the sensibility in this test that a good bolometer usually does, namely about 2,500 mm. for a deflection from a candle at a meter's distance. Yet it must be borne in mind that no bolometer will give a maximum of sensibility with a galvanometer of sensibility of only 4×10^{-10} amperes, yet in the case of the thermopile we have a deflection of 2,340 mm. with this galvanometer for the sensibility in terms of the "candle at a meter's distance."

In the above comparison the thermopile was much more steady in its action than the bolometer. Both possessed drift due to temperature changes, but the thermopile lacked the unsteadiness peculiar to the bolometer. The thermopile was, on the other hand, somewhat sluggish, a feature which the bolometer rarely possesses. Yet the sluggishness was not so pronounced as to be annoying.

In conclusion it may be said that this particular type of thermopile has overcome the usual objections applied to thermopiles in general, namely lack of sensibility and sluggishness. On the other hand, it possesses a sensibility superior to the best bolometers and lacks the annoying unsteadiness of the bolometer due chiefly to batteries. It possesses a slow drift, which likewise is common to the bolometer, and is more sluggish than the bolometer.

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PRINCETON UNIVERSITY,
May, 1910.

SOME CALORIMETRIC APPARATUS.

BY WALTER P. WHITE.

EXPERIMENTAL developments are often analogous to the progress of organic evolution. New structures and processes are derived from those already in use, even where an independent departure would be more appropriate, and features, once adopted, tend to continue long after the original reasons for them have ceased to exist. In calorimetry, the simple forms of apparatus which were suited to the early approximate methods have been applied to more exacting work by means of elaborate corrections and precautions, sometimes extending to the control of the observing room for hours previous to the actual observation, yet without attaining more than moderate accuracy. Several investigators have recently introduced new types of apparatus, with complete inclosure of the calorimeter, which are not only more accurate but on the whole simpler. Further developments and devices, largely inspired by this new departure, have issued in rapid succession from various laboratories. In two former papers of this series[1] there has been described a system in which definiteness of conditions was carried further by a modification of the calorimeter cover, and by attention to evaporation and the errors connected with stirring. Considerable simplification of calculation and an increase in thermometric accuracy are also secured. The present papers deal with the apparatus actually used, and the accuracy attained.

The apparatus was designed, and has been used, for the determination of specific heats, especially at high temperatures, by dropping bodies from heaters into the calorimeters.

1. *The Two Calorimeters.*—The two calorimeters are shown in

[1] PHYS. REV., 31, 545 and 562, 1910. References to these will be by page number simply, since they virtually form one article with the present papers.

section in Fig. 1. A holds 1.5, B, 3.8, liters. In the case of each the form was chosen mainly with reference to satisfactory stirring.

The type represented by B, with central inner tube, has the widest and shortest possible path for the water, and, therefore, presumably, the smallest heat production for stirring of given effectiveness.[1] In some respects it would be more satisfactory if symmetrical, but this would require some kind of gearing to run the stirrer, since a shaft coming centrally through the top is usually out of the

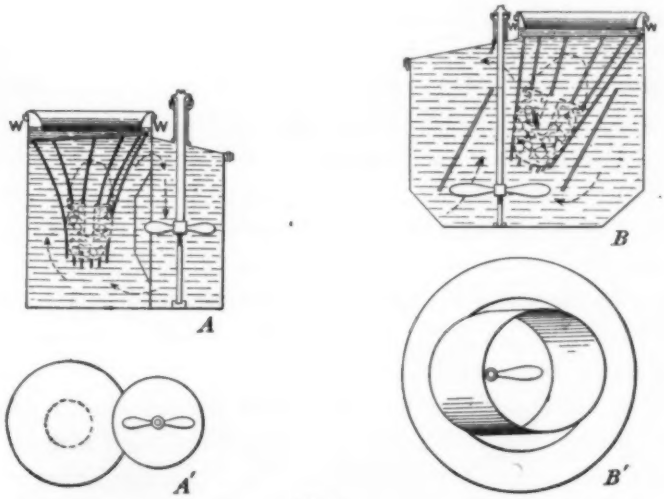


Fig. 1.

(One sixth actual size.) Calorimeters A and B, vertical section; A' and B', horizontal section. Features: Cover in contact with the water; prevention of evaporation; wide passages for water in circulating; absence of dead spaces. W, water level. The position of the crucible is shown by the dotted lines. Capacity, A, 1,500 c.c.; B, 3,800 c.c.

question. The inclined tube avoids this difficulty and gives almost all the advantages of the symmetrical form, yet with simple stirring arrangements. By inclining the whole calorimeter the circulation would become more nearly symmetrical; this arrangement seems almost necessary in adapting the type to smaller sizes. The propellers are removable. If the partition in A is removable, the whole interior is easily reached in cleaning. For the same reason the inner tube in B should be supported from the bottom, or made removable if possible.

[1] Cf. page 576.

The stirring of A has been shown[1] to be satisfactory. When B is stirred so as to have lags of about the same value, the heat produced is only .6 as much as with A, in spite of the greater size. The cooling factor for small temperature differences is .0027 for A, .0017 for B.

The light (17-gram) floating cover, originally used with calorimeter A, and described three years ago, has been given up for reasons stated on page 574. The present cover (70 grams—the same one is used for both calorimeters) touches the water, but rests on a ground joint. The annular space under the cover permits the water to vary somewhat in amount without either overflowing or failing to wet the cover; while the weight of the cover prevents it from being floated out of place when the water is high. An approximate preliminary adjustment of the amount of water is of course necessary, and is very easily obtained. Evaporation through the (dry) joint is about 7 mg. an hour, which is not likely to produce appreciable error unless the extra space in the chamber is very small. An oil seal gives very little trouble, and has been used where maximum accuracy was desired. A still more satisfactory arrangement seems possible.

2. *The Jacket.*—Complete inclosure of the calorimeter chamber, quick opening and closing, and thorough circulation without much heating, were the main requirements. In Fig. 2 the jacket is shown in section and also in top view, partly open. Water is held up in the two halves of the cover, and in the upper section of the chamber wall, by atmospheric pressure. On leaving the propeller the water divides and passes across through the three upper passages, and then returns through the lower space. Its circulation is directed by the partitions, P and Q, of which Q runs nearly the whole length of the tank outside the chamber, as shown by the dotted line. The chamber is opened by moving the covers aside; their downturned ends then move in the troughs left at the ends of the tank. The covers slide upon the stout rod T. The pulley for the calorimeter stirrer, and the whole jacket stirrer, are borne on one half cover. This half is clamped firmly in place during an observation; shoving aside the other fully exposes the calorimeter opening.

[1] Cf. page 575.

The pulley is connected to the propeller shaft of the calorimeter by a small fiber rod. This separable pulley removes from the calorim-

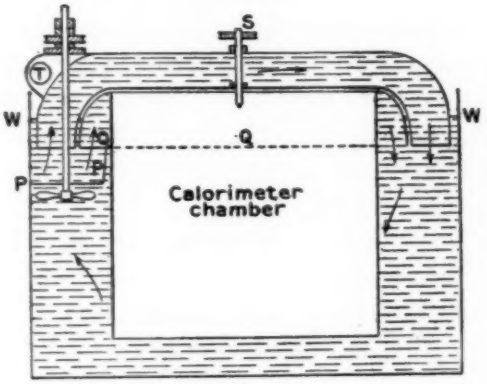


Fig. 2*A*.

(One sixth actual size.) Calorimeter jacket, giving complete inclosure by a single body of water, with free circulation of the water, and readily opened. Vertical section (empty).

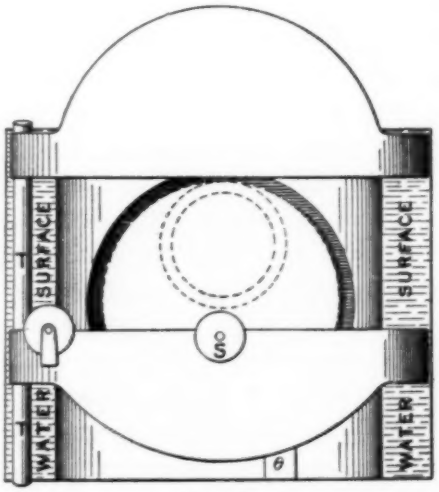


Fig. 2*B*.

Top view, opened to show the calorimeter (dotted lines) within. *T*, rod on which the top slides in opening. *S*, pulley which operates the calorimeter stirrer. *θ*, groove for passage of thermoelectric thermometer. *W*, water level.

eter the heat produced in the bearing, keeps the condition of the bearing constant in spite of temperature changes in the calorimeter,

and avoids running a belt into the chamber. Some metal workers may find difficulty in soldering the whole apparatus together, but this difficulty will disappear if the calorimeter chamber and the upper passage next it are made in a separate piece, and then fastened in place. The outer wall then becomes a simple rectangular box. The partitions are merely to guide the circulation, so tight soldering is needed only on the covers, the chamber, the upper inclosure around it, and the outside tank.

In an earlier jacket[1] the water entered and left the single cover by two downward projecting tubes on the same side, and the cover rotated about one of these to open the chamber. The advantages of the newer form are the wider and shorter path for the circulating water, and the ability to open the chamber freely and at the same time have thermometers, stirrers, etc., inserted in the calorimeter from above through the stationary part of the cover. The older form is preferable where it is desirable to expose the whole calorimeter quickly.

3. *The Crucible.*—A platinum crucible was necessary to hold the loose lumps of material generally employed, and was also used, as already pointed out,[2] to eliminate the errors due to *all* the heat losses in transferring from the furnace to the calorimeter. This was done by making determinations on the crucible alone, and taking for the effect of the substance investigated the difference between these results and those when the crucible was loaded. The crucible is shown in section in Fig. 3; the bail, of platinum wire, is for suspension; the reëntrant tube for the thermoelement fits very loosely around the porcelain insulating jacket tube, and so drops promptly away from it at the proper time. Some tests seemed to show that if the sides of the porcelain jacket were partly ground away for 2 cm. from the end, so as to expose the thermoelement to direct radiation from the crucible, the thermoelement

[1]Represented in simplified form in Amer. Jour. Sci., *28*, 337, 1909.

[2]Amer. Jour. Sci., *28*, 336, 1909. T. W. Richards and F. G. Jackson have since laid considerable emphasis on the improvement introduced into calorimetry by this method of "adiabatic transfer." (Z. für physik. Chem., *70*, 426, 1909.) The first one to use and report it appears to have been N. Stücker (Sitzber. kais. Akad. Wiss., Wien, Math.-nat. Klasse, *114*, 658, 1905), who mentions it as a means of eliminating radiation losses.

lag was diminished, and furnace regulation thus favored, but the equilibrium temperature was not altered.

In the earlier determinations, explosions of steam during cooling in the calorimeter blew most of the charge out of the crucible, and in one case projected some of it out of the calorimeter altogether. Although emptying a granular charge out into the free water circulation shortens the time for attaining temperature equilibrium, and may be recommended for that purpose, in the present case it was thought undesirable, for the charge needed re-weighing, which necessitated a heating and rather tedious cooling. With the present locking cover shown in the figure, the silicate charges only lose from 2 to 6 mg. at each cooling. More determinations can be thus made in a day, as a weighing once a day is now sufficient, for which the cooling may be done over night.

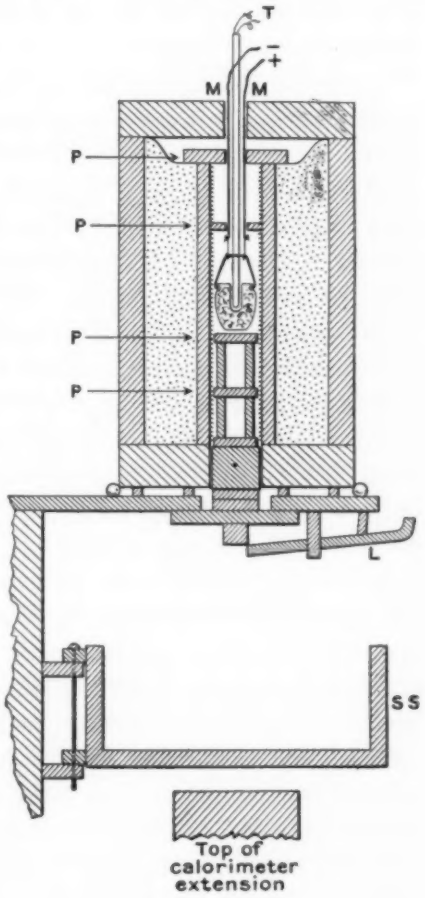


Fig. 3.

(One sixth actual size.) Platinum-wound furnace, used from 500° to 1500°. *P*, partitions to shield against the cooling effect of the ends of the furnace; *SS*, swinging shield; *L*, latch for dropping out the furnace bottom; *MM*, heavy wires for the dropping current. T. thermoelement.

4. *The Furnace.* — The high temperature furnace is of the type first used in this laboratory,[1] with internal platinum winding. This feature not only permits the attainment of a higher tem-

[1]Day and Allen, PHYS. REV., *19*, 177, 1904; Day and Clement, Amer. Jour. Sci., *26*, 411, 1908. See also Amer. Jour. Sci., *28*, 474, 1909.

perature, but at lower temperatures also has the decided advantage, not so often noted, of diminishing the lag between coil and chamber, and so increasing the quickness and the exactness with which a constant temperature may be reached and maintained. The furnace is 32 cm. high by 20 cm. in diameter; internal cavity, 4.8 cm. diameter.

In transferring the crucible to the calorimeter, the middle of the furnace bottom is first dropped away, and then, as soon after as possible, the crucible is dropped by an electrical arrangement.

The furnace (Fig. 3) is mounted on a stout iron plate, 5 mm. thick, with a 5 mm. heat-insulating air space between. An opening 5.5 cm. in diameter is cut through the furnace bottom and plate. This is closed by a plug of fire-clay, borne on a movable iron plate 10 cm. square. This is held up against the large plate by the latch *L* in such a way, readily intelligible from the figure, that a quick pull on the latch causes the plug to fall without tipping. Upon the plug, and inside the furnace chamber, is a pedestal, made up of three fire-clay partitions 6 mm. thick, each faced on both sides (except next the crucible) with reflecting discs of platinum foil, and supported by a light frame made by grinding away as much as possible from a thin porcelain tube 4 cm. in diameter. The pedestal has two functions. It protects the crucible from the cooling effect of the furnace bottom, and so makes for uniformity of temperature; and it supports the crucible during the hour or more necessary to reach and establish a constant and uniform temperature. Platinum is very soft at high temperatures, and the fine wires, which support the crucible for the interval of one second which elapses between the fall of the pedestal and that of the crucible, would have to be undesirably large if they were to hold for any length of time. At the higher temperatures the uppermost partition is apt to stick to the bottom of the crucible, hence it is tied by platinum wire to the plate below, whose weight is enough to pull it away. Above the crucible another platinum covered shielding partition is suspended. The crucible is fastened by fine wires to the larger wires, *MM*. The drop is made by melting the fine wires electrically, securing both quickness and simplicity of manipulation. This arrangement, borrowed from J. A. Harker,[1] has needed modification

[1] J. A. Harker, Phil. Mag., *10*, 430, 1905.

at the higher temperatures used in our work, first by the addition of the supporting pedestal, as already explained, second by precautions against the formation of an arc between the large wires as the small ones melt. Some arcing is necessary, for *both* sections of the fine wire must be melted; hence the current must not stop when the first goes. By using two separate, vertical, fine wires as far apart as possible, good results were usually obtained. The two small arcs first formed were then pulled out by the fall of the crucible. But at 1300° and over, where arcing is easier, and the fine wires have to be relatively heavy, even this method often fails. A better arrangement, therefore, will be reported later.

5. *The Swinging Shield.*—When the pedestal falls from the furnace it is caught in a light wooden box. This box, which can turn on a vertical pivot, is immediately swung aside by the observer's hand; the single quick motion performs the following operations: It carries the pedestal out of the way, shuts off the furnace current, pushes a temporary cover off from the calorimeter, and closes the circuit which drops the crucible. Immediately returned to its place, the box shields the calorimeter from the direct heat of the furnace, after an exposure of only a second for the drop. The heat radiated to the calorimeter is, at 1500°, 0.5 calorie per second, or 1/50,000 the quantity of heat then measured; hence a doubling of the interval would produce no appreciable error.

6. *The Splash Cover.*—The water which is thrown out of the calorimeter as the crucible enters it causes no error, for the weighing is done after the heat determination is finished. The slight resulting heat loss is one of the things eliminated by dropping the crucible alone. But even a very small amount of water falling on the outside of the calorimeter and evaporating there will generally cause a very serious error. The water raised by the splash, therefore, is confined by a removable upward extension of the calorimeter wall in the form of a heavy brass collar 11 cm. high, which presses a rubber gasket against the top of the calorimeter. The few drops which get over the top of this tube are caught by a metal pan. This splash cover is put in place some time before the drop, and is removed by hand immediately after it. The crucible and jacket are then covered for the first time. The method used to

compute the cooling correction (see page 547) permits this rapid and convenient procedure to be employed without disadvantage.

7. *The Temperature Measurements.*—The copper-constantan thermoelements used to measure the calorimeter temperature, the platinum-rhodium elements for the furnaces, and the potentiometer installation, have already been described.[1] The latter has been modified to suit the demands of the present work. The potentiometer is arranged as a double potentiometer,[2] that is, two separate switch settings can be made at the same time, either of which can, by a single motion of an exchanging switch, be brought into play along with the appropriate thermoelement. Other thermoelements, also, can be connected to the galvanometer alone, for direct reading of small electromotive forces. One switch setting is used to read the furnace temperature, the other, the calorimeter temperature. The difference between calorimeter and jacket is read by the galvanometer alone before the drop of the crucible, and afterward on the connection previously used for the furnace. The small differences within the furnace are, where necessary, read by direct deflection.

The whole system is fully equal to the demands of the present work. Just before the drop, four different readings are often made and recorded in half a minute. A small auxiliary potentiometer is used, in series with the main potentiometer, in order to provide an additional dial. The resulting combination enables satisfactory results to be obtained with a potentiometer originally of too small capacity for the present work.

The exchanging switch is quite different from that described in the earlier article, and is briefly referred to in the PHYSICAL REVIEW, Vol. 30, p. 784, 1910. Connections outside the switch are made by means of the wooden clothespin clamps described in the same article.

An electric dial, operated by impulses from a seconds pendulum, gives bell signals at appropriate intervals in each minute, and saves all necessity of looking for the time, either in reading temperatures or in dropping the crucible.

[1]Constancy of Thermoelements, PHYS. REV., *23*, 449, 1906; Potentiometer Installation, ibid., *25*, 334, 1907; The Thermoelement as a Precision Thermometer, ibid., *31*, 135, 1910.

[2]Zeitschr. f. Instr., *27*, 218, 1907.

8. *Protection Against Leakage in the Potentiometer System.*—The system of equipotential leakage shields described in a former article[1] has been extended. It was there pointed out that when a feeble current from an outside source leaks through a measuring circuit, the only detrimental part of it is that which goes through the galvanometer, and that this part can be diminished by (1) improving insulation; (2) so arranging the circuit that the vagrant current shall tend to pass through it by a path which does not include the galvanometer; (3) interposing a metallic conductor, the shield, between the circuit and the outside E.M.F. If an external shield is complete the protection it gives is perfect. But in a hot furnace, where the air conducts, a shield, to be complete, must be a complete inclosure, and this is usually inconvenient. There will, therefore, often be a small leakage current, and it is desirable to make this leave the circuit as near the furnace as possible, in order to fulfill condition (2). Yet it is obviously disadvantageous to provide any more or better paths for the vagrant current. The difficulty is avoided by having a second, internal, shield, connected to the circuit, but insulated from the external shield. It intervenes between the external shield and the potentiometer, switch, auxiliary resistances and battery. Such a shield can easily be insulated better than most ordinary apparatus, hence by its use leakage is diminished, yet whatever leakage does occur will leave the circuit by the most desirable path. The same shield may serve for protection against leakage in the storage battery circuit of the potentiometer.[2] In the installation here described, it is directly under the top of the observing table, where this rests on the legs, and consists merely of four (connected) plates of metal, 10 cm. square, set on four smaller insulating discs of paraffined hard rubber, which by their situation are well protected from dust and light. The external shield for the table is the metal legs. (In many cases it might conveniently be four small metal plates between the table legs and the floor.) In order to give protection from battery leakage, the shield must separate from the galvanometer circuit all apparatus which is

[1]Potentiometer Installation, *loc. cit.*, p. 340.

[2]*Loc. cit.*, p. 343. The present shield is merely a slight extension of the shield first used for the storage battery circuit.

near the positive potential of the battery. Hence all such apparatus rests on a metal plate, which is connected to those below the table. But no further insulation is required for this plate.

Recently, in humid summer weather, appreciable leakage was detected between the two sides of the thermoelement circuit, which differed only by .013 volt. It is impossible to shield against such leakage; the only effective defense is better insulation. Accordingly, a change was made in the installation. The exchanging switch was kept dry by inclosure in a tight, glass-topped case in which calcium chloride was placed. The leads passed out through sheaths of celluloid imbedded in paraffin, and all other exposed insulating surfaces, including the top of the potentiometer itself, were paraffined after being dried in the radiation from an incandescent light. These precautions have proved effectual in very humid weather. In some potentiometers now being constructed in the shop of the Geophysical Laboratory, provision is made for keeping them dry by an upper glass cover and the use of drying material. Such inclosure is in some ways more favorable to constancy than immersion in an oil bath, and is certainly cleaner. It seems likely to prove a valuable general method where high precision is desired along with freedom from uncertainty and troublesome precautions, especially in humid climates.

9. *Averaging.*—The important temperatures are all obtained by taking the mean of several observations. A short method of averaging is employed. As the separate observations are recorded, each is subtracted from the preceding. The time required for this is almost inappreciable, since the differences are nearly constant for one set of readings. Five readings are usually made, at half minute intervals. These will yield four differences. If these differences, in order, are called A, B, C, D, it is easy to show that the arithmetical mean of the five readings differs from the middle one by $[A-D+2(B-C)]/5$, a value which is almost never more than one unit of the last place, and can be obtained at a glance. Similar formulæ can of course be obtained for three or seven readings.

10. *Conduct of an Observation.*—While the furnace is heating, the calorimeter is filled, its temperature regulated (by ice or by an immersed incandescent light) if necessary, the ice tank is filled,

and the splash cover is put in place. When the desired furnace temperature has been reached and held long enough to heat the crucible uniformly, the thermoelement in the furnace is removed, and the accurate element used for the final measurement is inserted instead. After about three minutes, readings of the furnace temperature can be resumed, and are alternated with readings of the calorimeter temperature. As soon as constancy of temperature in the furnace is again assured, the drop is made. This requires five simple operations. At a bell signal from the clock the latch is pulled, letting the pedestal fall. Immediately after a quick glance to see if this has happened properly, the box is swung out and back, dropping the crucible. The splash cover is then removed, the calorimeter and jacket are covered, and the observer returns to the galvanometer telescope, arriving in ample time for the first reading, fifteen seconds after the drop.

During the next two or three minutes the temperature rise is rapid, and readings are made every fifteen seconds. One-minute readings then begin, and in the intervals between them the preparation for the next determination is started. The porcelain tube and crucible attachments are removed from the furnace to get cool, and the furnace is closed again to keep hot. At the end of the temperature observations, about ten minutes later, the crucible can be at once adjusted for another heating. Meanwhile the reduction of the observations has also been nearly completed.

11. *The Lead Bath.*—It was found that in the furnace described above, the absolute, and still more the relative, uniformity of temperature over the crucible is a maximum at the highest temperature measured (1500°) and decreases continuously down to 500°, where the extreme difference is about 15°. Hence for work at 300° and 500° a furnace of a different type was made, designed so as to give more uniform temperatures, and shown in section in Fig. 4. The bottom, a soft steel annular plate, is joined to the two soft steel concentric vertical tubes, *A* and *B*, by beading the metal of the tubes over the edges of the bottom. The annular space between the tubes is filled with a lead-tin alloy which is stirred by a propeller, with water-jacketed bearing. The heating coil is in three sections, at top, bottom, and sides. These are separately regulated, in ac-

cordance with the readings of differential thermoelements, so as to give a temperature uniform to 3° or 4° all over the outside. The bottom swings aside to expose the crucible for dropping. In swinging it trips a catch which releases the crucible.

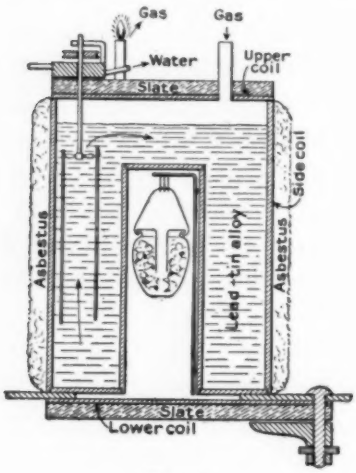


Fig. 4.

(One sixth actual size.) Electric furnace, for temperatures not only constant but very uniform, from 180° to 600°. Of steel, filled with a lead and tin mixture, which is stirred, and is kept from oxidation by means of illuminating gas.

Actual measurements showed that when the bottom of the chamber was 10° warmer than the center, the resulting temperature difference on the surface of the crucible was not over 0.5°. When the bottom of the chamber was 10° cooler, the effect on the crucible was imperceptible. When equilibrium has once been established, the temperature of the bath seldom attempts to vary faster than about 1° in fifteen minutes. This bath has several times been raised to 600°.

A slow stream of illuminating gas keeps the melted metal free from oxide.

12. *The Steam Oven.*—Values of the mean specific heat up to 100° are obtained with the apparatus shown in Fig. 5. This differs from some others of its kind in that boiler and crucible chamber are combined in one vessel. It is thus simple, compact, and manageable, takes little heat, and avoids all difficulties from throttling or imperfect circulation of the steam. It is made of sheet copper, soldered with tin. When in use, it is over the calorimeter; the small flame keeps the water inside boiling just fast enough so that condensation continues in the air condenser, K; the asbestos board partitions G, G, keep the flame from interfering with the temperature either of the crucible chamber or of the calorimeter. A shield, S, guides the flow of the steam inside. A copper-constantan thermoelement measures the temperature of the chamber. Two hours are allowed for the charge to attain the temperature of the steam.

Direct observations with a thermoelement in the middle of the charge showed that at least 90 minutes is necessary if results are to be accurate to .1°. The partitions, of cork covered with tin foil, in the lower part of the chamber, protect the crucible from the cooling effect of the bottom.

13. *Dropping the Crucible from the Steam Oven.*—In dropping from the steam oven, it is possible to get rid of the splash and splash cover. To accomplish this, the crucible is lowered rapidly by a

Fig. 5.

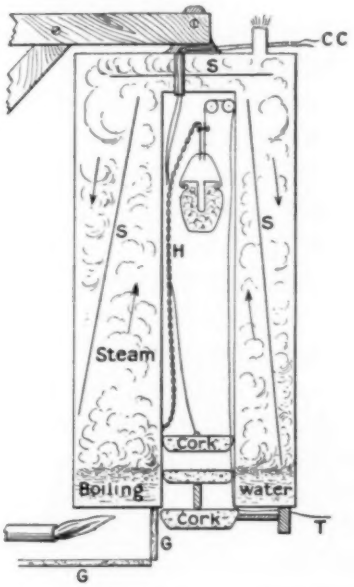
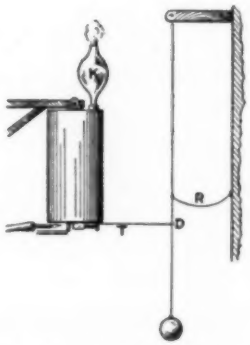


Fig. 6.



(One sixth actual size.) Steam oven for temperature of 100°. *S*, radiation shield; *cc*, copper constantan thermoelement; *H*, chain; *GG*, asbestos guards around the flame.

Diagrammatic view illustrating the automatic method of dropping the crucible from the steam oven of Fig. 5.

cord, checked just before it strikes the water, and at once released automatically. The automatic release is performed by a pair of brass tongs with two handles, one of which closes, the other opens them. To one handle the cord is attached, to the other a chain, *H*, whose other end is fastened to the inside of the oven chamber. Hence the crucible, raised by the cord, is held firmly, but drops as soon as it is lowered enough to pull upon the chain. A constant

heat loss in the drop is furthered by also using an automatic arrangement (Fig. 6) to lower the crucible. This is hardly necessary at such low temperatures, but simplifies the manipulations, and so diminishes accidents, while the labor of installing it is the merest trifle. Its essential feature is a pendulum consisting of a heavy ball hung by a cord. This, near the middle of its swing, is restrained by another cord, *R*, and then swings about its attachment as a center. The cord which lowers the crucible is attached a little below at *D*; thus the motion of the crucible changes suddenly from a high to a low velocity as the cord, *R*, tightens; this happens just before the crucible touches the water. The pendulum is first released by the pull of a cord attached to the jacket cover, which, here taking the place of the swinging shield above described, is thrown back to make the drop. The motion of the cover is also made to release a stretched rubber band, which jerks a temporary cover away from the calorimeter. The upper cork partition in the oven chamber is attached to the chain, and leaves the chamber only when the crucible does. It protects the crucible from appreciable heat loss after the chamber is opened.

The operation of dropping, then, comprises four motions, among which confusion is very unlikely, first (at leisure) opening the chamber; second, at the clock signal, moving the cover, when the drop follows automatically, and then covering calorimeter and jacket as usual.

SUMMARY.

This paper deals with a calorimetric system in which all conditions affecting the calorimeter temperature are, as far as possible, definite and measured, and in which as a result errors once serious are made negligible, computation is often simplified, and an increase in effective thermometric accuracy can be secured. The apparatus was designed for the determination of specific heats, especially at high temperatures, by the mixtures method.

1. Two calorimeters are described, designed for effectiveness in stirring. In one of them the use of an *inclined* central tube affords a path both wide and short for the circulating water, and at the same time allows the stirrer itself to remain simple. Copper covers, carefully fitted and in contact with the water, practically eliminate evaporation, both externally and internally.

2. When in their jacket these calorimeters are surrounded above and on all sides by a single compact body of circulating water, and have, therefore, a uniform environment. The calorimeter chamber can be opened at any time without disturbing either the circulation or the connections to the calorimeter.

3. The internally-wound type of furnace commonly used in the Geophysical Laboratory, modified to permit the dropping of bodies, was used for most of the temperatures employed. At 1500° the temperature of the charge was found uniform to a small fraction of one per cent.

4. A light swinging shield under the furnace controls the exposure of the calorimeter. It performs with certainty, ease and quickness the functions often sought through a movable calorimeter or furnace.

5. Temperatures uniform and constant to 0.1° at least are obtained for the interval from 200° to 600° in an electrically heated and stirred bath of melted metal, which surrounds but does not touch the body to be heated.

6. A steam-heated oven is also used. With this a simple automatic arrangement gives uniformity of results in dropping bodies into the calorimeter.

7. This paper makes a short addition to a previous discussion of equipotential leakage shields.

8. By partly inclosing the apparatus in cases which also contain drying material, and then coating with paraffine all exposed insulating surfaces, insulation was assured, without inconvenience, under atmospheric conditions previously considered almost fatal to high accuracy in the measuring system.

GEOPHYSICAL LABORATORY,
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WASHINGTON, D. C., July 16, 1910.

A TEST OF CALORIMETRIC ACCURACY.

BY WALTER P. WHITE.

IF an adequate storage battery and an accurate signal clock are available, the electrical method of calorimeter calibration is convenient, accurate, and reliable. In the case now to be described the measurements were varied so as to give, besides a calibration, some information as to the different sources of error present.

I. APPARATUS AND METHODS.

As electrical calibrations have already been carefully described,[1] such features of apparatus and method as are not new may be given very briefly. The calorimeters were *A* and *B* of the previous article.

1. *Electrical Energy Measurements.*—The heater was of the type introduced by Jaeger and von Steinwehr. Its resistance was about 20 ohms, maximum intended capacity, 5 amperes at 100 volts, or about 7,000 calories a minute. As usual, the current and voltage were read on a potentiometer by means of resistances so adjusted that the two readings were approximately the same, and could be taken from the same potentiometer setting. They were made on the same potentiometer as was used for the thermoelectric temperature measurements. The maximum value of each was about 15,000 microvolts. The storage battery was often constant and steady to .00002 or better for the time of a (four-minute) heating. This steadiness, when attained, required an unexpected amount of attention to contacts, and an unexpectedly long previous exercising of the storage battery.

The standards were tested six times during the work, and one of these tests was a calibration at the National Bureau of Standards. The tests indicated that no error as great as 0.1 per mille was ever

[1] W. Jaeger and H. von Steinwehr, Ann. d. Phys., *21*, 23, 1906. H. C. Dickinson, E. F. Mueller and E. B. George, Bull. Bur. Standards, *6*, 379, 1910.

present in any of the resistances. The agreement of the current and voltage measurements made during each heating was, as far as computed, better than this in all but a few determinations.

2. *Prevention of Leakage.*—The low potential end of the heating circuit was in contact with calorimeter and jacket through the case of the heater. The equipotential shield[1] under jacket and measuring system was therefore kept at about the same potential by a direct connection to the negative current terminal. Leakage to the jacket at 100 volts pressure through the insulation of the positive lead of the heater was still possible, even if not probable. This would have produced an error, since the leakage current, after reaching the jacket, would have gone along the negative lead and so through the current measuring resistance, but would not have gone through the heater. This possibility was absolutely prevented by a small shield, which completely separated the lead from the jacket, being insulated from both, and was connected to the main shield beyond the current measuring resistance. Any current leaking out of the lead so as to pass the coil was thus guided past the resistance also.

3. *Time Measurement.*—The heat communicated to the calorimeter is a function of time as well as of watts. The time was given by a chronometer with electric contact, which was rated and found to have far more than the necessary accuracy. With the four-minute runs used, an accidental variation of 0.01 second in the electric signal would have meant an error of only 0.04 per mille in the final result.

4. *The Automatic Time Switch.*—In some preliminary work, a chronograph used to compare between the heating current and the clock proved somewhat unsatisfactory. The marking pens needed attention just when the observer was particularly busy elsewhere. A device whereby the clock signal itself turns on the heating current has been a source of great satisfaction. It not only requires less attention, and is more certain, but it has a smaller probable error, and gives the time in even intervals, without the inconvenience of any correction.[2] It is shown, diagrammatically, in Fig. 1. Two

[1]See p. 679.

[2]The idea of making the clock turn the current on and off was borrowed from Dr. N. E. Dorsey of the National Bureau of Standards. His apparatus, however, was different.

wooden arms, A and B, each 12 cm. long, swing, one above the other, on the rod, R, as a center. Each is pulled toward the other by a spring. A carries a projecting copper rod, W, B has a copper plate, P, which can make contact with W and also with a fixed rod, V, underneath A. A can be drawn back and held by the catch, l, controlled by the magnet, M, which is operated by the clock. The plate on B is then drawn against V, making contact. A, when released, is pulled forward so that W suddenly strikes the plate, making contact, and at the same instant driving the plate away from

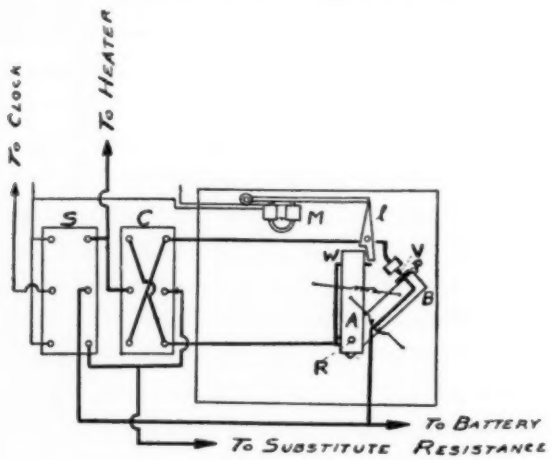

Fig. 1.

Automatic time switch for starting and stopping the heating current by the clock signals.

V. The plate is connected to the +pole of the battery; V and W are connected, one to the heater, the other to the substitute resistance, which is employed (as usual) to bring the battery to equilibrium for the current needed. These connections are made through the commutator, C, and can be exchanged. Hence at the beginnng of a heating the apparatus is used to turn the current from the substitute resistance to the heater, and at the end to change back again. In each case the change occurs the same fraction of a second after the magnet impulse. In parallel with this apparatus is one side of the two-pole, two-throw switch, S. The other side is so arranged in connection with the magnet, M, that when the switch is

closed either way the circuit is closed through M, but when the switch is open the magnet is dependent on the clock, and responds to its signals. The switch is operated as follows: Most of the time the current flows through S. C is open and the apparatus beyond it is dead. Some time before the beginning or end of a heating, A is pulled back and caught by the catch, l. One second previous, the observer closes C on the proper side, putting the automatic switch in parallel with S, and then opens S, leaving the other in circuit alone. The clock now opens the circuit through the magnet, M, causing the automatic switch to operate. Immediately, the observer closes S on the other side, and then opens C. The automatic switch has started (or stopped) the current in the heater exactly on time, but it has only been in use about half a second, and an excess resistance of 2 ohms in it would have produced no appreciable error.

5. *Weights.*—The calorimeter was weighed with the water, and the oil for the seals then added from a pipette whose delivery was constant to 15 mg. The weighing was to about 30 mg. or .00002, with calorimeter A, and to greater proportional accuracy with B. The weight taken was always the same for each calorimeter. The final water equivalents of the calorimeters were computed from the mean value of the results. In fact, the finding of these equivalents was the first object of the investigation. Heat quantities obtained with these calorimeters will therefore really be referred to the electrical units. The (distilled) water was at first usually boiled and cooled to free it from air, but no difference was observed in the results when this precaution was omitted, and in the latest (and best) determinations it was never employed. A direct observation of the amount of gas given off on heating one sample of the unboiled water to 100° indicated that there should have been no error from liberation of gas for the temperature range, about 22° to 42°, used in the work.

6. *Temperature Measurements.*—The temperatures were measured essentially as described in the preceding article, p. 678. Successive readings of the final temperature were concordant to 0.1 microvolt (0.00033°), the smallest unit read or used; those of the differential temperature (ϕ of p. 546) to about .002°.

7. *Conduct of an Observation.*—During the four-minute heating periods, four readings of the (varying) temperature, ϕ, and twelve

TABLE I.

Sample Record of a Determination. Observations in Roman type. Data given by calculation, in Italics. All explanatory matter added since the observations is in brackets.

[Time.]	June 10, 1910. Calorimeter A. Second 17-degree run. Cadmium [cell] temperature, 21.7°.				
	[Differential Temperature in Microvolts.]	[Calorimeter Temperature in Microvolts.]		[Mean Temperature.]	[Interval.]
	[ϕ]	[ϑ]	[$\delta\vartheta$]	[ϑ]	[$\Delta\vartheta$]
9		7091.9	.2		
		91.7	.2		
10		91.5	.1	*7091.5*	
		91.4	.2		
11		91.2			
12	66.7				*3.8*
18	66.9				
19	*33.6* [sum]	7088.0			
	66.8 [mean]	87.8	.2		
20		87.7	.1	*7087.7*	
		87.5	.2		
21	65.6				
22	[Current turned on]				
		[Current Reading]		[Voltage Reading.]	
	131 [=65.6×2]	14455.0			
	188				
				14430.2	
23		56.5			
		57.2			
	536			30.8	
24		57.6			
		57.8			
	890				
				30.8	
25		58.1			
		58.3			
	1252				
	5894 [=1473.5×4]			30.5	
		58.3			
	8891 [sums]	*58.8*		*2.3*	
	[means]	*14457.4*		*14430.6*	
26	[End of heating]				

[Time.]	June 10, 1910. Calorimeter A. Second 17-degree run. Cadmium [cell] temperature, 21.7°.				
	[Differential Temperature in Microvolts.]	[Calorimeter Temperature in Microvolts.]		[Mean Temperature.]	[Interval.]
		[ϑ]	[$\delta\vartheta$]	[Mean ϑ]	[$\Delta\theta$]
27	1479.9				
28	1473.5				
29		12711.6			
		700.5	*11.1*		
			11.3		
30		689.2		*12689.2*	
		78.0	*11.2*		
			11.3		
31		66.7			
32	1448.8				
38	1413.2				*218.9*
39	*2862.0* [sum]	12491.9			
	1431.0 [mean]		*10.7*		
		81.2			
			10.7		
40		70.5		*12470.6*	
			10.6		
		59.9			
			10.6		
41		49.3			

of the (nearly constant) electrical energy were made. These readings were more numerous than was necessary, but could be taken with a facility which was almost leisurely, thanks to the double potentiometer, the arrangements for constant galvanometer sensitiveness, and the short period (three seconds) of the galvanometer. Except in a few special determinations, all the periods were of ten minutes. This gave abundant time for the attainment of steady conditions after the four-minute heatings. The cooling periods were of the same length, ten minutes, which facilitated computation.

Table I. gives a sample record, showing the observations and calculations made during the 32 minutes occupied by a run. The corrections for variation from Newton's law of cooling, described on p. 553, were also usually looked up during the observation period.

The remaining calculations[1] are shown in Table II.

The small correction described on p. 557 was also applied to the final results.

8. *Plan of the Investigation.*—The determinations were varied in

[1]These calculations are simple enough, but seem worth giving in view of the emphasis that has recently been laid by several writers on the large amount of labor required for computing Regnault-Pfaundler cooling corrections.

TABLE II.

Remaining Computations.			*Actual work to the left; explanation to the right.*
			Cooling Correction Calculations.
218.6	1543.0	9329	Data, taken from previous table.
3.8	52.0	520	
214.8	*1491.0*	*8809*	Differences.
			Temperatures.
	7087.7		Lower temperature.
	381.0		Thermoelement correction.[1]
	12689.2		Upper temperature.
	3.8		Cooling correction.[2]
	126.9		
	12819.9		Corrected upper temperature.
	434.3		Thermoelement correction.[1]
	13254.2		Final upper temperature.
	7468.7		Final lower temperature.
	5785.5		Interval.
			Final Calculations.
	.06302		Log. calibration constant.[3]
	.31938		2×log. energy readings.
	.23766		Co-log. interval.
	.62006		Log. equivalent.
	4.1693		Equivalent.

two ways: (1) As to temperature interval. Whenever either calorimeter was filled and weighed it was sooner or later heated through the maximum interval used (from 17° to 15° for A, 9° for B), but this interval was usually divided into several steps, each of which formed a complete determination. The objects were: first, to see if the equivalent of the calorimeter varied with temperature, and second, to investigate the errors present, by comparing determinations representing a variety of conditions. (2) The time of heating was also increased to 14 minutes in some cases, so as to vary conditions still further.

[1]These corrections were taken from a table. The corrected results are in three-thousandths of a degree.

[2]The second number was obtained with a slide rule from the three differences above The correction is: $(\Delta\vartheta_2-\Delta\vartheta_{-1})\div(\phi_2-\phi_{-1})\times(\Phi_1-\Phi_{-1})+\Delta\vartheta_{-1}$. See pages 548–9.

[3]This calibration constant took in the value of the various coils used to measure the energy, the (temporarily assumed) water equivalent of the calorimeter, the rate of the clock, the (small) correction for varying temperature of the standard cadmium cell, etc.

The following determinations were made with calorimeter *A*: five sequences of two determinations each, giving five "lower nines" (from about 23° to 32°) and five upper nines (from about 32° to 41°); next, similarly, four sequences of three each, giving lower fives, upper fives, and four more upper nines; then two sequences of four each, giving six 1.3-degree intervals and two 15-degree intervals, and lastly, seven single 17-degree runs, in two of which the heating continued fourteen minutes.

With calorimeter *B* there were eight 9-degree determinations (the first two of which were rejected because they were vitiated by a wrong adjustment in the potentiometer system); alternating with these there were three sequences of 5-degree determinations, giving three upper and three lower fives.

Later, four sequences were run with calorimeter *A*, each of a 5-degree, followed by a 9-degree run. The weather was then hot, and the sequences started at about 29°.

II. Results.

The results are shown in Tables III., IV. and V. The equivalent given for each calorimeter is the number of joules needed to heat a gram of water one degree, computed by means of a water equivalent derived from the mean result of the calibration.

Remarks.—During the time covered by the first part of Table III., the atmospheric humidity was very high; this condition, of course, is apt to cause electrolytic leakage over the insulating surfaces. On the morning when the observations were begun, it was necessary to first stop and dry and paraffine the top of the potentiometer itself in order to eliminate a serious leakage error. The results of Table III. therefore represent the performance of the apparatus under unusually trying conditions.

After the work for Table III. was finished it was discovered that during part, or perhaps all, of it a leakage error of about 0.5 microvolt had been present. The whole installation was then gone over, and protected from the effects of moisture, as described on p. 680. Although the weather was no better, the results then obtained with calorimeter *B* (Table IV.) were much more concordant. In order to see if this was due to the improved protection against

TABLE III.

First Calibration of Calorimeter A. Insulation in the Potentiometer System somewhat Imperfect.

Series Number.	Date.	Interval.	Cooling Correction, parts per Mille.	Equivalent.	Mean of Series.	Deviation from Mean. Per Mille.	Average Deviation from Mean.	
							In Per Mille.	In Degrees.
1	May 21		22	4.1891		.05		
3	" 23	Lower	23	[4.1901]		—		
5	" "	9.3°	24	4.1891	4.1893	.05	.05	.00048°
7	" 26		24	96		.07		
9	" "		24	95		.04		
2	" 21		58	4.1897		.04		
4	" 23	Upper	58	97		.04		
6	" "	9.3°	59	94	4.1895	.03	.046	.00042°
8	" 26		54	97		.04		
10	" "		54	92		.08		
11	" 27		23	4.1886		.08		
14	" "	Lower	23	92		.06		
17	June 4	4.7°	23	94	4.1889	.11	.09	.00042°
20			23	85		.10		
12	May 27		55	4.1894		.17		
15	" "	Upper	55	82		.12		
18	June 4	4.7°	53	89	4.1887	.05	.11	.00052°
21	" "		53	83		.09		
13	May 27		57	4.1890		.04		
16	" "	Upper	57	95		.08		
19	June 4	9.3°	56	93	4.1891	.04	.06	.00055°
22	" "		56	88		.08		
23	" 6		15	4.1911		.59		
24	" "		15	4.1889		.06		
25	" "		43	79		.18		
27	" "	1.3°	45	67	4.1887	.47	.22	.00029°
28	" "		72	87		.01		
29	" "		74	86		.01		
26	" "		30	4.1889		.02		
30	" "	15°	31	90		.00		
32	" 9	18°	40	4.1891		.03		
33	" 10		43	90		.00		
31	" 9		21	4.1893	4.1890	.07	.03	.00052°
34	" 10		23	89		.02		
35	" "	17°	24	87		.07		
36	" "		24	91		.04		
37	" "		24	88		.03		

TABLE IV.

Calibration of Calorimeter B. Potentiometer Installation Improved.

Series Number.	Date.	Interval.	Cooling Correction, per Mille.	Equivalent.	Mean of Series.	Deviation from Mean. Parts per mille.	Average Deviation from Mean. In Per Mille.	Average Deviation from Mean. In Degrees.[1]
40	July 1	Lower 4.7°	15	4.1899		.03		
42	" 2		15	95	4.1895	.03		
48	" 5		13	92		.04	.05	.00023°
41	" 1	Upper 4.7°	35	4.1894		.08		
43	" 2		35	97	4.1896	.02		
49	" 5		33	97		.09		
38	June 30		—	[4.1881]		—		
39	" "		—	[4.1877]		—		
44	July 2		15	4.1884		.05		
45	" "	9.5°	16	80	4.1882	.04	.036	0.00034°
46	" "		15	80		.04		
47	" 5		15	83		.03		
50	" "		14	83		.03		
51	" "		14	81		.02		

TABLE V.

Second Calibration of Calorimeter A, with Improved Potentiometer Installation.

Series Number.	Date.	Interval.	Cooling Correction, per Mille.	Equivalent.	Mean of Series.	Deviation from Mean. Parts per mille.	Average Deviation from Mean. In Per Mille.	Average Deviation from Mean. In Degrees.[1]
52	July 11		26	4.1895		.02		
54	" "	Lower 4.7°	25	94	4.1896	.04	.03	.00014°
56	" "		25	97		.03		
58	" "		25	97		.03		
53	" "		42	4.1893		.03		
55	" "	Upper 9.3°	42	96	4.1894	.04	.02	.00019°
57	" "		43	94		.01		
59	" "		41	94		.01		

	Weight.	Water Equivalent.	Wt. of Water.	Total Equivalent.
Calorimeter *A*	647.79 gr.	62.78	1,412.21	1,474.99
Calorimeter *B*	1,364.17	127.62	3,778.05	3,905.67

[1] The actual temperature error is smaller than this number.

leakage, and also if there was any systematic time variation in the system, calorimeter *A* was again used, as shown in Table V. The results are even better than in Table IV., and so appear to show that much of the error in Table III. was due to leakage, and certainly show that results practically free from such error can now be obtained in spite of very trying conditions.

III. INTERPRETATION.

1. *The Cooling Correction.*—The largest temperature interval in these results is over thirteen times the smallest; in most of the series the calorimeter was heated about twenty degrees above its jacket; the majority of the shorter intervals began at or above the temperature at which such intervals would ordinarily end; the upper intervals have cooling corrections at least two and a half times as great as the lower, and the correction was also increased in a different way with the slow heating (nos. 32 and 33). Yet on comparing all the results there is no indication either of a systematic error or of an increase in the accidental errors, due to the cooling correction. Furthermore, the accidental errors common to all the intervals will soon be shown to arise mainly from the temperature measurement. Hence the error in the cooling correction is, for this work, considerably smaller than the accidental errors, and that is very much smaller than the errors of all work hitherto published in this field, whether done with a cooling correction or by methods in which such a correction is avoided. The present investigation thus gives a direct confirmation of the conclusion, derived over a year ago from an analysis of the calorimeter problem,[1] and presented more fully in the second paper of this series,[2] that all cooling correction errors are at present quite negligible with proper apparatus.

2. *Accidental Errors.*—With the cooling correction eliminated, the sources of error are four: Temperature, time, electrical energy and weight. The accidental errors in temperature measurement are probably nearly independent of the interval, and therefore produce a much larger proportional effect in the shorter intervals. Errors in time and weight, on the contrary, produce the same proportional

[1] PHYS. REV., *28*, 462, 1909.

[2] Pp. 576, 579.

effect throughout, and so tend to be most prominent in the larger intervals, because there the temperature errors are less important. For the same reason the accidental errors in energy measurement are probably most effective in the larger intervals, though they diminish somewhat as the interval increases. If, now, the proportional error is multiplied by the temperature interval, the result is the total error expressed in degrees. This quantity is tabulated in column 9 of the tables. Roughly speaking, it is equal to $E_\theta + nP$ if E_θ is the temperature error, n the interval, and P the other errors, taken as proportional to the heat quantity. Hence if the error were mainly due to temperature measurement, the quantity in column 9 would be about the same for intervals of different length, and its tendency to increase with the interval is a measure of the magnitude of the other errors. A rigorous application of this formula to the present data would of course be absurd on account of the natural and evident variations in the errors from one series to another. But the general increase in all the tables of the quantity in column 9 as the interval increases leaves no doubt that considerable (accidental) errors are present other than the temperature error. And, more specifically, there is very little doubt that in Tables IV. and V. the proportional part of the error (nP) was appreciably larger for the 9-degree than for the 5-degree interval, and hence, also, must have been appreciable for the 5-degree interval. The accidental temperature error alone, therefore, was for these tables almost certainly under .00020°. It therefore follows that:

(1) Since in the temperature measurements the smallest quantity read, 0.1 microvolt, corresponded to .00033°, or to nearly double the probable temperature error, the precision indicated for the thermoelectric thermometer is all that could possibly have been expected (see page 578).

(2) This temperature error, small as it is, predominates in the 5-degree interval results, and probably in the 9-degree also.

(3) Since that part of the accidental error not due to temperature may be partly or largely due to the energy measurement, the precision indicated for the calibration as a whole, namely, one part in 30,000 for a 9-degree interval, and one in 20,000 for 5 degrees, is merely an inferior limit for the precision of the calorimetric instal-

lation alone, and this precision may approach the limit set by the thermometric errors, or better than one part in 25,000 for 5 degrees, and one part in 50,000 for 10 degrees. In this case, of course, the preponderating error would be in all cases due to the temperature measurement. The work represented in Table III. was, as already stated, affected by a source of error which was afterward removed. But even these results show a precision more than half of that attained later, and far more than was considered possible in calorimetry until lately.

These indications as to the precision attainable and the relative importance of temperature errors corroborate the conclusions reached in preceding papers of this series, and in the abstract published last year. Clearly shown also is the advantage of the large temperature intervals which are a characteristic feature of the present calorimetric work. This result contradicts the generally prevalent opinion favoring small intervals, and fully justifies the departure from current practice which was made[1] three years ago in designing the present installation for large intervals.

That an increase from 10° to 20° would yield much further advantage with the present system is not clearly demonstrated by the present data. It certainly would with temperature measurements accurate only to .0005°.

There is little significance in the fact that the quantity of heat measured in many of the determinations is larger than usually employed by other experimenters. Calorimeter *B*, taking three times as much heat as calorimeter *A* for a given interval, happens to show larger errors. Wherever the accuracy varies with the quantity of heat, the essential factor is evidently the accompanying increase in the interval. For the same intervals the results would be nearly as accurate with considerably smaller quantities of heat. With more sensitive thermometers, however, the same accuracy could, of course, be obtained with smaller intervals.

3. *Systematic Errors.*[2]—If a calorimeter is given a direct calibration, the constant systematic errors of the calorimeter itself are neces-

[1] PHYS. REV., *25*, 137, 1907.

[2] This section was rewritten, after some supplementary observations in the fall of 1910, and was re-submitted October 22.

sarily eliminated for the conditions of each particular determination. By varying these conditions the systematic *variations* in the calibration factor can also be eliminated. Systematic errors in time measurement and in weighing seem safely negligible in the present case. The remaining possibilities seem to be systematic errors in the electrical standards; and slight variations due to temperature change in the potentiometer coils.

The precautions taken with the standards and the agreement obtained in the observations on them, as described in sec. 1, p. 686, appear to exclude the possibility of any systematic error here greater than one part in 10,000 (.1 per mille). Evidence as to the other systematic errors is given by the data in Tables III., IV., V. It there appears that, for calorimeter *A*, discrepancies, sometimes slightly exceeding .1 per mille, occur between the mean results of the different series, but these are not concordant; for instance, the value for the 9-degree intervals, which fell a little from May to June, returned again in July, and the 5-degree interval, lower than the 9-degree in June, was higher in July. There is therefore little doubt that these discrepancies, where not purely accidental, are due to minute variations in the potentiometer, or to the leakage mentioned on p. 693 which was probably greatest in June. It seems practically certain that they do not indicate any systematic source of error in the calorimeter itself, for they are connected with causes (change of weather, lapse of time) which can hardly have an effect on the calorimeter, while those factors that might be expected to affect it, such as the magnitude of the cooling correction, the mean rate of cooling, and the time of heating, produce an altogether imperceptible effect. Moreover, whatever systematic variations there may be in the specific heat of water or of the calorimeter metals at different temperatures, or in the calibration curve of the thermoelectric thermometer, the final resultant of all of them is imperceptible here.

With calorimeter *B*, on the other hand, the few, though concordant, 5-degree series differ by about .3 per mille from the 9-degree, though the last results with calorimeter *A*, only six days later, were concordant for the different intervals over about the same temperature range. This, of course, suggested the energy

measurement as the source of error, but a special investigation showed that the standard coils were never perceptibly heated by the greatest currents actually used, and that the potentiometer coils concerned had not enough irregularities to account for more than a fraction of the discrepancy observed.

In the following fall, some more calibration runs were made to test the matter further. The results were generally better than in Table IV., but were not essentially different; hence it seems unnecessary to give them in full. The mean values of four series were:

Interval.	Time of Heating.	Heating Current.	Cal. Factor.
5° lower	4 min.	3.5 amperes	4.1887
5° upper	4 min.	3.5 amperes	4.1887
9°	9 min.	3.5 amperes	4.1887
9°	4 min.	5.0 amperes	4.1890

The former discrepancy has now completely disappeared, and with it has gone the possibility of fully explaining it. Since these results are intermediate in value between the older ones, it seems probable that the former difference was *partly* accidental. The absence of variation with time of heating or magnitude of interval again indicates the absence of error due to the calorimeter itself. The slightly higher value of the fourth series may be accidental; in this series the heating current was unusually unsteady.

Unless, therefore, some important sources of error have been overlooked, the final systematic errors now seem little, if any, greater than 0.2 per mille. This is several times the error of the calorimeter (with the thermometer) alone, but is more than sufficient for the work for which the installation is to be used.

Relative Value of the Results.—A comparison with the work of others is rather difficult, for most of the best work hitherto has been done with mercury thermometers sensitive only to .001° and with small intervals, which of course precludes any approach to the accuracy of which the calorimeter itself is capable. The tenfold greater precision of the present determinations is not far from what should result from the improvement in conditions, and there appears to be no reason why it should not be readily attained wherever the accuracy of the associated determinations is sufficient to make it worth while.

Summary.

The two calorimeters described in the previous paper were calibrated by the electrical method, and the determinations were varied considerably, so as to afford information as to the errors present.

An automatic switch, which started and stopped the heating current in obedience to electric signals from a chronometer, was an advantageous substitute for a chronograph; otherwise, the details of the calibration were much as usual.

The results were:

1. The average accidental error of a complete calorimetric determination has been reduced to about one part in 30,000. The essential conditions for this performance appear to have been: Ordinary care in manipulation and observing, an electrical thermometer reading consistently to .0003°, auxiliary electrical apparatus accurate to. 00003, a complete and uniform inclosure for the calorimeter, regulation of the stirring, elimination of evaporation, and the use of unusually large temperature intervals. Of these features, only the last two are specially characteristic.

2. The average accidental temperature error was brought below .00025°. This is practically a perfect performance for the multiple thermoelement used as a thermometer.

3. No error due to the cooling correction could be detected, even where the total average error was only .00003.

4. The preponderant accidental calorimetric error was in the temperature measurement.

5. No systematic error due to the calorimeter could be detected. The constancy of the auxiliary electrical apparatus was not great enough to bring the systematic error of the final result of the calibrator below 0.1 per mille (.0001).

6. The use of a large temperature interval obviously diminishes the effect of the temperature error. It brings no corresponding disadvantage. Intervals much larger than those ordinarily used are therefore to be recommended where accuracy is desired.

Geophysical Laboratory,
Carnegie Institution of Washington,

July 16, 1910.

NEW BOOKS.

Introduction to Physical Chemistry. By HARRY C. JONES. New York: The Macmillan Company, 1910. Pp. xv + 279. Price, $1.60.

Les Compteurs Électriques. By L. BARBILLION and G. FERROUX. Paris: Gauthier-Villars, 1910. Pp. vii + 226. Price, 3.25 fr.

Electric Motors: their Action, Control and Application. By FRANCIS B. CROCKER and MORTON ARENDT. New York: D. Van Nostrand Company, 1910. Pp. vi + 291. Price, $2.50.

Solenoids, Electromagnets and Electromagnetic Windings. By CHARLES R. UNDERHILL. New York: D. Van Nostrand Company, 1910. Pp. xix + 342. Price. $2.00.

Principles and Methods of Geometrical Optics. By JAMES P. C. SOUTHALL. New York: The Macmillan Company, 1910. Pp. xxiii + 626. Price $5.50.

Populär-Wissenschaftliche Vorlesungen. By E. MACH. Leipzig: Johann, Ambrosius Barth, 1910. Pp. xii + 508. Price, M. 6.80.

Theorie der Kinematographischen Projektionen. By KARL MARBE. Leipzig: Johann Ambrosius Barth, 1910. Pp. 80. Price, M. 2.40.

Der Lichtbogen als Wechselstromerzeuger. By K. WILLY WAGNER. Leipzig: S. Hirzel, 1910. Pp. iv + 119. Price, M. 3.60.

Geschichtstafeln der Physik. By FELIX AUERBACH. Leipzig: Johann Ambrosius Barth, 1910. Pp. iv + 150. Price, M. 4.

Téléphonie. By ALBERT TURPAIN. Paris: Gauthier-Villars, 1910. Pp. 186. Price, 5 fr.

Notions Fondamentales sur La Télégraphie. By ALBERT TURPAIN. Paris: Gauthier-Villars, 1910. Pp. 180. Price, 5 fr.

Elementary Crystallography. By W. S. BAYLEY. New York: McGraw-Hill Book Company, 1910. Pp. xii + 241.

The Construction of Graphical Charts. By JOHN B. PEDDLE. New York: McGraw-Hill Book Company, 1910. Pp. viii + 109.

Elementary Mechanics. By F. M. HARTMANN. New York: John Wiley and Sons, 1910. Pp. xi + 119.

The Foundations of Alternate Current Theory. By C. V. DRYSDALE. London: Edward Arnold, 1910. Pp. vii + 300.

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ERRATA.

Page 611. In the abstract of the paper by O. W. Richardson on Gravitation and the Electron Theory, two terms of the equation on the third line, referred to in the text, were omitted in printing. The two paragraphs on page 611 should read as follows:

These conditions are satisfied if the repulsion between two elements δE, $\delta E'$ of electric charge is of the form:

$$\frac{\delta E \delta E'}{r^2} F\left(\frac{dM}{dE}, \frac{dM'}{dE'}\right) = \frac{\delta E \delta E'}{r^2}\left[1 + a\frac{dM}{dE} + b\frac{dM'}{dE} + c\left(\frac{dM}{dE}\right)^2 + d\left(\frac{dM}{dE}\,\frac{dM'}{dE'}\right) + e\left(\frac{dM'}{dE'}\right)^2\right]$$

where dM and dM' are the infinitesimal amounts of mass associated with dE and dE' respectively, a, b, c, d, and e are constants, and F denotes some functional relation. A formula of this type might be expected to arise if the usual law of force were modified to a slight extent by the disturbance in the ether in the immediate neighborhood of the element of charge, since, according to certain forms of electrical theory, the mass of any region is proportional to the energy confined in that region.

The constants a, b, c and e do not enter into the expression for the force between two neutral particles. They would, however, make a difference in the effect of gravity on an electric charge. Experiments on electrons may possibly indicate such a difference. There seems to be no reason why gravitation should not act differentially on opposite charges. An action of this kind might account for the magnetic fields of the earth and the heavenly bodies.

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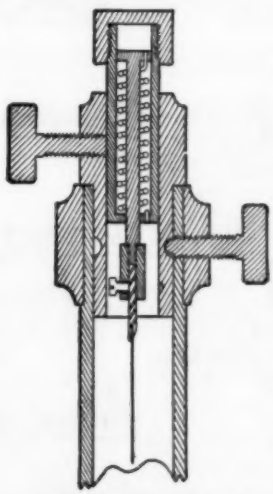

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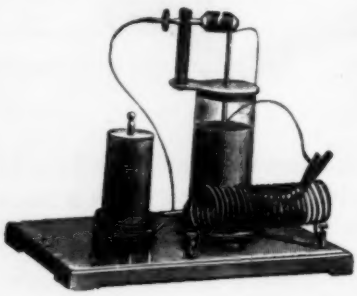

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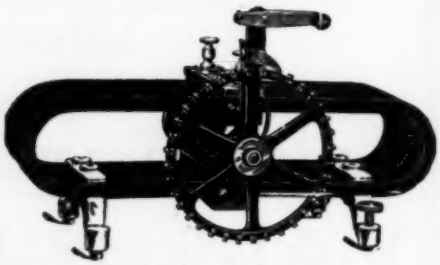

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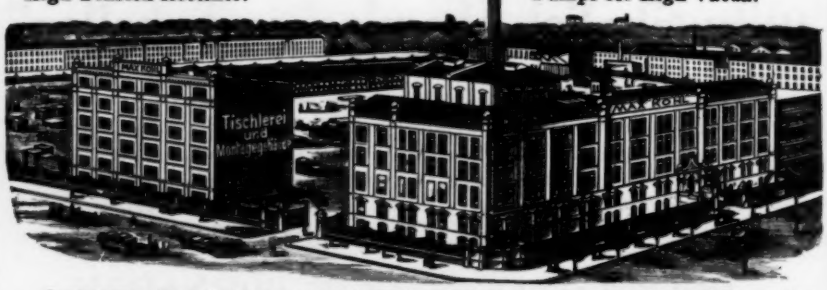

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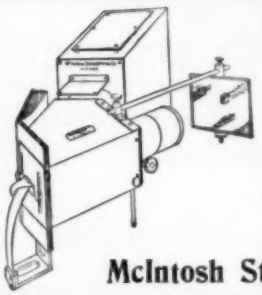

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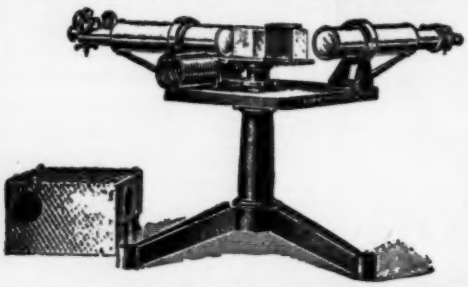

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